Unit F2

Differentiation

Introduction

In Unit D4 *Continuity* you studied *continuous* real functions and saw that they share some important properties; for example, they all satisfy the Intermediate Value Theorem, the Extreme Value Theorem and the Boundedness Theorem. However, many of the most interesting properties of functions are obtained only when we further restrict our attention to *differentiable* functions.

You have already met the idea of *differentiating* a given real function f; that is, finding the gradient of the tangent to the graph $y = f(x)$ at those points of the graph where a tangent exists. The gradient of the tangent at the point $(c, f(c))$ is called the *derivative* of f at c, and is written as $f'(c)$. In this unit we investigate which real functions are differentiable, and we discuss some of the important properties that all differentiable functions possess.

Many of the techniques of differentiation and the properties of differentiable functions covered in this unit will not be new to you, but our main focus here is on establishing a rigorous foundation for these ideas. We will make frequent use of results on limits and continuity that you met in previous analysis units.

1 Differentiable functions

In this section we define what it means for a real function f to be differentiable at a point c, and we establish that certain basic functions are differentiable and find their derivatives. We also consider functions which possess higher derivatives, that is, functions which can be differentiated more than once. Finally, we show that differentiable functions are continuous, and also prove that the blancmange function, which was shown to be continuous in Unit F1 *Limits*, is in fact not differentiable at any point in its domain.

1.1 What is differentiability?

Differentiability arises from the geometric concept of the *tangent* to a graph. The tangent to the graph $y = f(x)$ at the point $(c, f(c))$, if it exists, is the line through the point $(c, f(c))$ whose direction is the 'limiting direction' of the chords joining the points $(c, f(c))$ and $(x, f(x))$ as x tends to c. This idea is illustrated in Figure 1.

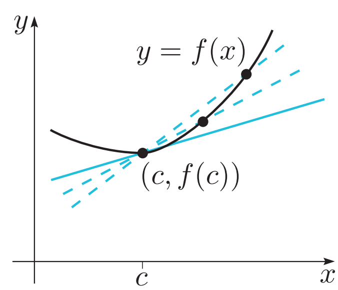
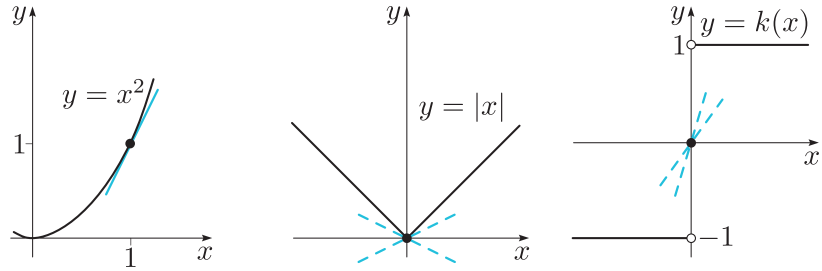


Figure 1 Constructing the tangent to a graph

The three examples shown in Figure 2 below illustrate some of the possibilities that can occur when we try to construct tangents in particular instances.



(a) (b) (c)

Figure 2 Different possibilities that occur when trying to construct tangents (the rule for the function k is given in the text)

The function

$$f(x) = x^2 \quad (x \in \mathbb{R})$$

is continuous on $\mathbb{R}$, and its graph (Figure 2(a)) has a tangent at each point; for example, the line $y = 2x - 1$ is the tangent to the graph at the point $(1, 1)$.

The function

$$g(x) = |x| \quad (x \in \mathbb{R})$$

is also continuous on $\mathbb{R}$, but its graph (Figure 2(b)) does not have a tangent at the point $(0, 0)$: no line through the point $(0, 0)$ is a tangent to the graph. This is because the 'limiting direction' of the chords described earlier is different depending on whether $(0, 0)$ is approached from the left or from the right. However, there is a tangent at every other point of the graph.

Finally, the function

$$k(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0, \end{cases}$$

is discontinuous at 0, and no line through the point $(0, 0)$ is a tangent to the graph (see Figure 2(c)). However, there is a tangent at every other point of the graph. We now make these ideas precise, by using the concept of limit (which you studied in Unit F1) to pin down what we mean by 'limiting direction'. We define the **gradient** (or slope) of the graph at $(c, f(c))$ to be the limit, as x tends to c, of the gradient of the chord through the points $(c, f(c))$ and $(x, f(x))$. The gradient of this chord is

$$\frac{f(x) - f(c)}{x - c}, \quad \text{where } x \neq c,$$

as illustrated in Figure 3. This expression is called the **difference quotient** for f at c. Thus the gradient of the graph of f at the point $(c, f(c))$ is

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}, \qquad (1)$$

provided that this limit exists. We call this limit the *derivative* of f at c.

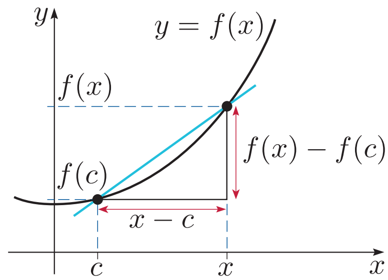


Figure 3 The chord joining $(c, f(c))$ and $(x, f(x))$

Sometimes it is more convenient to use an equivalent form of the difference quotient. If we replace x by $c + h$, then '$x \to c$' in expression (1) is equivalent to '$h \to 0$'. The difference quotient for f at c is then

$$Q(h) = \frac{f(c+h) - f(c)}{h}, \quad \text{where } h \neq 0,$$

as illustrated in Figure 4, and the gradient of the graph of f at $(c, f(c))$, that is, the derivative of f at c, is given by

$$\lim_{h \to 0} Q(h),$$

provided that this limit exists.

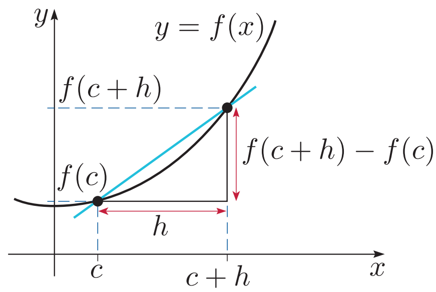


Figure 4 The chord joining $(c, f(c))$ and $(c + h, f(c + h))$

To formalise this concept, we need to ensure that f is defined near the point c, so we assume that c lies in some open interval I in the domain of f.

Definitions

Let f be defined on an open interval I, and let $c \in I$. Then the **derivative** of f at c is

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

that is,

$$\lim_{h \to 0} Q(h), \quad \text{where} \quad Q(h) = \frac{f(c+h) - f(c)}{h},$$

provided that this limit exists. If the limit exists, then we say that f is **differentiable at c**. If f is differentiable at each point of its domain, then we say that f is **differentiable** (on its domain).

The derivative of f at c is denoted by $f'(c)$ and the function $f' : x \longmapsto f'(x)$ is called the **derivative**, or sometimes the **derived function**, of f.

The operation of obtaining $f'(x)$ from $f(x)$ is called **differentiation**.

Remarks

1. The word 'differentiable' arises because the definition involves the *differences* $f(x) - f(c)$ and $x - c$.
2. The above box defines the notation for derivatives that we will normally use in this module, but there are several others in common use. Sometimes f' is denoted by Df and $f'(x)$ is denoted by $Df(x)$. Another frequently used notation is Leibniz notation, in which $f'(x)$ is written as $\dfrac{dy}{dx}$, where $y = f(x)$.
3. Note that we require f to be defined on an open interval containing c because, by the definition of the limit, x can approach c along any sequence of points in a punctured neighbourhood of c. In the next subsection we will define one-sided derivatives using the corresponding one-sided limits that you met in Unit F1.
4. The existence of the derivative $f'(c)$ is not quite equivalent to the existence of a tangent to the graph $y = f(x)$ at the point $(c, f(c))$. If $\lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c}$ exists, then the graph certainly has a tangent at the point $(c, f(c))$, and the gradient of the tangent is the value of the limit. However, the converse is not necessarily true. The graph may have a *vertical* tangent at the point $(c, f(c))$, in which case

$$\frac{f(x) - f(c)}{x - c} \to \infty \text{ (or } -\infty) \quad \text{as } x \to c.$$

In this case, $\lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c}$ does not exist, so f is not differentiable at c.

The idea of finding the gradient of a graph which is not a straight line was one of the first steps in the development of calculus. The major figures in this development were Sir Isaac Newton (1642–1727) in England and Gottfried Wilhelm Leibniz (1646–1716) in Germany, to whom we owe the 'dy/dx' notation and the names 'differential calculus' and 'integral calculus'. The names 'derived function' and 'derivative' and the notation f' were introduced much later by Lagrange (1736–1813). Another mathematician who played a role in the early development of calculus was Pierre de Fermat (1601?–1665) in France.

Worked Exercise F9

Prove that the function $f(x) = x^3$ is differentiable at each point $c \in \mathbb{R}$, and determine $f'(c)$.

Solution

To prove that a function is differentiable, it is usually most convenient to use the $Q(h)$ form of the difference quotient.

The difference quotient for f at c is

$$\begin{aligned} Q(h) = \frac{f(c+h) - f(c)}{h} &= \frac{(c+h)^3 - c^3}{h} \\ &= \frac{(c^3 + 3c^2h + 3ch^2 + h^3) - c^3}{h} \\ &= 3c^2 + 3ch + h^2, \quad \text{where } h \neq 0. \end{aligned}$$

Thus $Q(h) \to 3c^2$ as $h \to 0$, so f is differentiable at c, with $f'(c) = 3c^2$.

Here we have used the Combination Rules for limits without mentioning them explicitly.

We use c to denote a particular point where we are testing for differentiability. However, when stating the rule of a derivative, we replace c by the usual variable x. Thus the derivative of f in Worked Exercise F9 is $f'(x) = 3x^2$.

To prove from the definition that a function is *not* differentiable at a point, we need to show that the limit of the difference quotient does not exist. The following strategy is based on Strategy F1 from Unit F1.

Strategy F5

To prove that a function is not differentiable at a point, show that $\lim_{h \to 0} Q(h)$ does not exist by doing either of the following.

- Find two null sequences (h_n) and (k_n) with non-zero terms such that the sequences $(Q(h_n))$ and $(Q(k_n))$ have different limits.
- Find a null sequence (h_n) with non-zero terms such that $Q(h_n) \to \infty$ or $Q(h_n) \to -\infty$.

The next worked exercise illustrates how to apply this strategy.

Worked Exercise F10

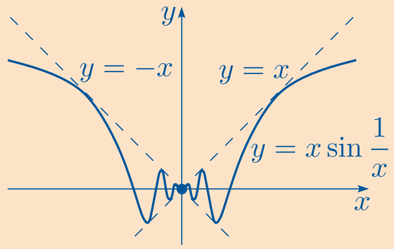
Prove that the function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is not differentiable at 0.

Solution

The graph of f is shown below.



The difference quotient for f at $c = 0$ is

$$\begin{aligned} Q(h) &= \frac{f(h) - f(0)}{h} \\ &= \frac{h \sin(1/h) - 0}{h} = \sin(1/h), \quad \text{where } h \neq 0. \end{aligned}$$

Since $\sin(1/h)$ oscillates infinitely often in any interval containing the origin, giving different tangents at different points, we use the first method of Strategy F5.

Consider the two sequences

$$h_n = \frac{1}{n\pi} \quad \text{and} \quad k_n = \frac{1}{\left(2n + \frac{1}{2}\right)\pi}, \quad n = 1, 2, \ldots.$$

Then (h_n) and (k_n) are null sequences with non-zero terms, chosen so that

$$Q(h_n) = \sin(1/h_n) = \sin(n\pi) = 0 \to 0 \text{ as } n \to \infty$$

and

$$Q(k_n) = \sin(1/k_n) = \sin\left(2n + \tfrac{1}{2}\right)\pi = 1 \to 1 \text{ as } n \to \infty.$$

Since these limits are different, f is not differentiable at 0.

Other choices of (h_n) and (k_n) are possible; for example,

$$h_n = \frac{1}{2n\pi} \text{ and } k_n = \frac{1}{\left(2n - \frac{1}{2}\right)\pi}, \quad n = 1, 2, \ldots,$$

giving $Q(h_n) = 0$ and $Q(k_n) = -1$.

Worked Exercise F10 shows that the domain of a derivative f' can be smaller than the domain of f.

Exercise F15

(a) Prove that the function $f(x) = 1/x$ is differentiable at each point $c \in \mathbb{R} - \{0\}$, and determine $f'(c)$.

(b) Prove that the function

$$f(x) = \begin{cases} x^2 \cos(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is differentiable at 0, and determine $f'(0)$.

Hint: Use the Squeeze Rule for limits from Subsection 1.3 of Unit F1.

(c) Prove that the function $f(x) = |x|$ is not differentiable at 0.

(d) Prove that the function

$$f(x) = \begin{cases} |x|^{1/2} \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is not differentiable at 0.

We now establish the derivatives of some basic functions.

Theorem F21 Basic derivatives

(a) If $f(x) = k$, where $k \in \mathbb{R}$, then $f'(x) = 0$.

(b) If $f(x) = x^n$, where $n \in \mathbb{N}$, then $f'(x) = nx^{n-1}$.

(c) If $f(x) = \sin x$, then $f'(x) = \cos x$.

(d) If $f(x) = \cos x$, then $f'(x) = -\sin x$.

(e) If $f(x) = e^x$, then $f'(x) = e^x$.

Proof (a) If $f(x) = k$, then the difference quotient for f at any point is 0, so $f'(x) = 0$.

(b) The difference quotient for f at c is

$$Q(h) = \frac{f(c+h) - f(c)}{h} = \frac{(c+h)^n - c^n}{h}.$$

Recall that, for $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + b^n.$$

This is the Binomial Theorem, which you met in Subsection 3.4 of Unit D1 *Numbers*.

So, by the Binomial Theorem,

$$\begin{aligned} Q(h) &= \frac{1}{h}\left(c^n + nc^{n-1}h + \frac{n(n-1)}{2}c^{n-2}h^2 + \cdots + h^n - c^n\right) \\ &= nc^{n-1} + \frac{n(n-1)}{2}c^{n-2}h + \cdots + h^{n-1}, \quad \text{where } h \neq 0. \end{aligned}$$

Thus $Q(h) \to nc^{n-1}$ as $h \to 0$, so f is differentiable at c for any $c \in \mathbb{R}$, and $f'(x) = nx^{n-1}$.

(c) The difference quotient for f at c is

$$\begin{aligned} Q(h) &= \frac{\sin(c+h) - \sin c}{h} \\ &= \frac{\sin c \cos h + \cos c \sin h - \sin c}{h} \\ &= \cos c\left(\frac{\sin h}{h}\right) + \sin c\left(\frac{\cos h - 1}{h}\right), \quad \text{where } h \neq 0. \end{aligned}$$

You met the limits

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h\to 0}\frac{\cos h - 1}{h} = 0$$

in Subsections 1.1 and 1.3 of Unit F1, respectively.

Thus

$$Q(h) \to \cos c \times 1 + \sin c \times 0 = \cos c \;\text{ as } h \to 0,$$

so $f'(x) = \cos x$, as required.

(d) The proof of this part is similar to that of part (c). We omit the details – you may like to write out the proof for yourself.

(e) The difference quotient for f at c is

$$\begin{aligned} Q(h) &= \frac{e^{c+h} - e^c}{h} \\ &= e^c\left(\frac{e^h - 1}{h}\right), \quad \text{where } h \neq 0. \end{aligned}$$

You saw that $\lim_{h\to 0}\frac{e^h - 1}{h} = 1$ in Subsection 1.3 of Unit F1.

Thus

$$Q(h) \to e^c \times 1 = e^c \;\text{ as } h \to 0,$$

so $f'(x) = e^x$, as required. ■

In general, when we differentiate a function f, we obtain a new function f' whose domain may be smaller than that of f. The notion of differentiability can then be applied to the function f', yielding another function whose domain consists of those points where f' is differentiable.

Definitions

Let f be differentiable on an open interval I, and let $c \in I$. If the derivative f' is differentiable at c, then we say that f is **twice differentiable at c**, and the number $f''(c) = (f')'(c)$ is called the **second derivative of f at c**. The function f'', also denoted by $f^{(2)}$, is called the **second derivative** (or **second derived function**) of f.

Similarly, we can define the **higher-order derivatives** of f, denoted by $f^{(3)} = f'''$, $f^{(4)}$, and so on.

Remarks

1. You may meet many different ways of denoting the second derivative of a function. For example, f'' is sometimes denoted by Df' or $D^2(f)$.

   In Leibniz notation $f''(x)$ is written as $\dfrac{d^2y}{dx^2}$, where $y = f(x)$.
2. Some functions can be differentiated as many times as we like. For example, if $f(x) = e^x$, then

   $$f'(x) = e^x, \quad f''(x) = e^x, \quad f^{(3)}(x) = e^x, \quad \ldots .$$

   However, not every derivative is differentiable at all points of its domain; you will see an example of this in Exercise F16.

1.2 One-sided derivatives

Although we know that the function $f(x) = |x|$ is not differentiable at 0, the graph of f shown in Figure 5 suggests that chords which join the origin $(0, 0)$ to points $(h, f(h))$ have gradients equal to 1 if $h > 0$, and equal to -1 if $h < 0$. This example suggests the concept of a *one-sided derivative*.

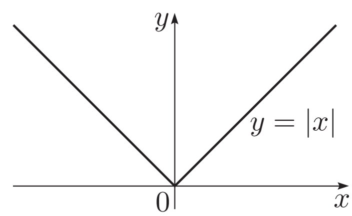


Figure 5 The graph of $f(x) = |x|$

Definitions

Let f be defined on an interval I, and let $c \in I$. Then the **left derivative** of f at c is

$$f'_L(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0^-} Q(h),$$

provided that this limit exists. If the limit exists, we say that f is **left differentiable** at c.

Similarly, the **right derivative** of f at c is

$$f'_R(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0^+} Q(h),$$

provided that this limit exists. If the limit exists, we say that f is **right differentiable** at c.

Remarks

1. The definition uses the concept of right and left limits which were defined in Subsection 1.4 of Unit F1.
2. In some texts, a function f defined on a bounded closed interval I is *defined* to be differentiable on I if f has a derivative at each interior point of I and appropriate one-sided derivatives at each endpoint of I.

The next theorem establishes the relationship between derivatives and one-sided derivatives.

Theorem F22

Let f be defined on an open interval I, and let $c \in I$.

(a) If f is differentiable at c, then f is both left differentiable and right differentiable at c, and
$$f'_L(c) = f'_R(c) = f'(c).$$

(b) If f is both left differentiable and right differentiable at c, and $f'_L(c) = f'_R(c)$, then f is differentiable at c and
$$f'(c) = f'_L(c) = f'_R(c).$$

Theorem F22 is closely related to the next theorem, the Glue Rule for differentiable functions, which is illustrated in Figure 6 and stated below. We omit the proofs of both these results as the ideas involved are very similar to those used in the proof of the Glue Rule for continuous functions (see Subsection 2.2 of Unit D4).

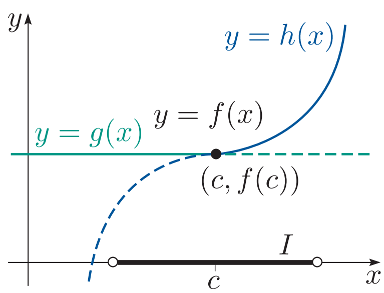


Figure 6 The Glue Rule for differentiable functions

Theorem F23 Glue Rule for differentiable functions

Let f be defined on an open interval I, and let $c \in I$. If there are functions g and h defined on I such that

1. $f(x) = g(x)$, for $x \in I$, $x < c$,
   $f(x) = h(x)$, for $x \in I$, $x > c$,
2. $f(c) = g(c) = h(c)$, and
3. g and h are differentiable at c,

then f is differentiable at c if and only if $g'(c) = h'(c)$. If f is differentiable at c, then $f'(c) = g'(c) = h'(c)$.

The Glue Rule enables us to determine whether or not certain hybrid functions are differentiable at particular points, without using the definition of differentiability. The next worked exercise illustrates how to use the Glue Rule in this way.

Worked Exercise F11

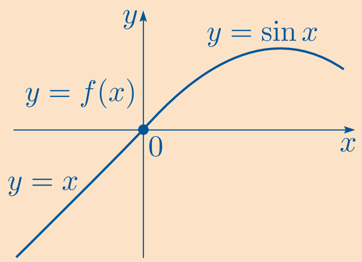
Use the Glue Rule to prove that the function

$$f(x) = \begin{cases} x, & x < 0, \\ \sin x, & x \geq 0, \end{cases}$$

is differentiable at 0, and determine $f'(0)$.

Solution

The graph of f is shown below.



Let I be the open interval $\mathbb{R}$ and define

$$g(x) = x \quad (x \in \mathbb{R}) \quad \text{and} \quad h(x) = \sin x \quad (x \in \mathbb{R}).$$

Then f is defined on I and $0 \in I$. Also,

$$\begin{aligned} f(x) &= g(x), \quad \text{for } x < 0, \\ f(x) &= h(x), \quad \text{for } x > 0, \end{aligned}$$

so condition 1 of the Glue Rule holds with $c = 0$.

Furthermore, $f(0) = g(0) = h(0) = 0$, so condition 2 holds, and g and h are differentiable with

$$g'(x) = 1 \quad \text{and} \quad h'(x) = \cos x,$$

so condition 3 holds.

Since $g'(0) = 1 = h'(0)$, we deduce that f is differentiable at 0, with $f'(0) = 1$, by the Glue Rule.

If we want to prove that the function f in Worked Exercise F11 is differentiable at a point c other than 0, then we can use the fact that differentiability is a *local property* of the function; that is, it depends on the behaviour of the function in any open interval (no matter how short) containing c. Thus

$$f'(x) = \begin{cases} g'(x) = 1, & x < 0, \\ h'(x) = \cos x, & x > 0, \end{cases}$$

and hence, on combining this result with Worked Exercise F11, we have

$$f'(x) = \begin{cases} 1, & x \le 0, \\ \cos x, & x > 0. \end{cases}$$

You can use this approach in the following exercise.

Exercise F16

Prove that the function

$$f(x) = \begin{cases} -x^2, & x < 0, \\ x^2, & x \ge 0, \end{cases}$$

is differentiable, and has derivative $f'(x) = 2|x|$.

Since the function $f'(x) = 2|x|$ is not differentiable at 0, Exercise F16 shows that a derivative need not be differentiable at all points of its domain.

Another consequence of the fact that differentiability is a local property is that the *restriction* of a differentiable function to an open subinterval of its domain gives a new differentiable function. For example, the function

$$f(x) = x^2 \quad (x \in (2,3))$$

is differentiable, since $f(x) = x^2$ is differentiable on $\mathbb{R}$.

1.3 Continuity and differentiability

Next we discuss the relationship between continuity and differentiability. First we show that a differentiable function is continuous.

Theorem F24

Let f be defined on an open interval I, and let $c \in I$. If f is differentiable at c, then f is continuous at c.

Proof If f is differentiable at c, then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c).$$

For $x \in I$ and $x \ne c$, we have

$$f(x) - f(c) = \frac{f(x) - f(c)}{(x - c)} \times (x - c).$$

Hence, by the Combination Rules for limits (see Subsection 1.3 of Unit F1),

$$f(x) - f(c) \to f'(c) \times 0 = 0 \quad \text{as } x \to c.$$

Thus $f(x) \to f(c)$ as $x \to c$, so f is continuous at c. ■

The following corollary gives us a test for non-differentiability; it is simply the contrapositive of Theorem F24 (and so is equivalent to it).

Corollary F25

Let f be defined on an open interval I, and let $c \in I$. If f is discontinuous at c, then f is not differentiable at c.

For example, the function

$$k(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0, \end{cases}$$

is discontinuous at 0 because $\lim_{x \to 0^+} k(x) = 1 \neq k(0)$. Thus k is not differentiable at 0, by Corollary F25.

It is important to remember that a function can be continuous at a point but *not* differentiable at that point; for example, the modulus function is continuous at all points of $\mathbb{R}$, but it is not differentiable at 0, as you saw in Exercise F15(c). This example can readily be modified to produce a continuous function which is not differentiable at any given finite set of points. It is even possible for a function to be continuous throughout its domain but *nowhere* differentiable, as we discuss below.

A continuous nowhere-differentiable function (optional)

The remainder of this section is not assessed, and is included only for your interest.

In the nineteenth century, when the concepts of continuity and differentiability were first made precise, it was widely believed that if a function is continuous at all points of an interval, then it must be differentiable at most points of that interval. However, it turns out that there exist functions which are continuous everywhere but differentiable *nowhere*. The first example was found as early as 1834 by Bernard Bolzano (1781–1848), but his pioneering work on analysis was not widely known. The first well-known example was constructed by Karl Weierstrass (1815–1897) in 1872. Such ‘pathological’ functions were regarded by some with suspicion. For example, the French mathematician Charles Hermite (1822–1901) wrote to a colleague in 1893, ‘I recoil in fear and loathing from that deplorable evil: continuous functions with no derivatives.’ However, in modern times it has been shown that such functions are in some sense normal and even useful.

In Subsection 3.2 of Unit F1 you met the *blancmange function* and saw that it is continuous at all points in its domain $\mathbb{R}$. We now prove that this function is nowhere differentiable. Recall that the blancmange function B is defined as follows:

$$B(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} s(2^n x) = s(x) + \tfrac{1}{2}s(2x) + \tfrac{1}{4}s(4x) + \cdots \quad (x \in \mathbb{R}), \tag{2}$$

where s is the sawtooth function

$$s(x) = \begin{cases} x - \lfloor x \rfloor, & 0 \leq x - \lfloor x \rfloor \leq \frac{1}{2}, \\ 1 - (x - \lfloor x \rfloor), & \frac{1}{2} < x - \lfloor x \rfloor < 1. \end{cases}$$

The graph of the function s has a 'corner' at each point of the form $k/2$, where $k \in \mathbb{Z}$, so the graph of each function $x \longmapsto 2^{-n}s(2^n x)$ has a corner at each point of the form $k/2^{n+1}$, where $k \in \mathbb{Z}$. This suggests that the graph of the function B, shown in Figure 7, does not have a tangent at any point of $\mathbb{R}$, and we now prove that this is the case.

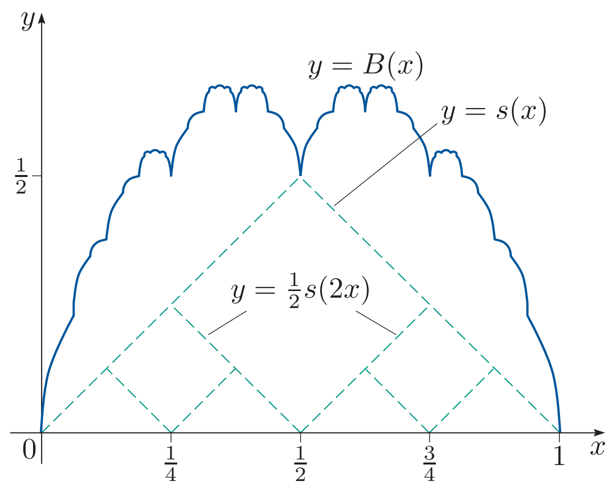


Figure 7 The blancmange function

The name 'blancmange function' was used in the 1980s by the English mathematician David Tall, who remarked that B is nowhere differentiable because it wobbles too much!

Theorem F26

The blancmange function is not differentiable at any point $c \in \mathbb{R}$.

To prove Theorem F26, we first prove three preliminary lemmas.

As we will need to refer to the difference quotients of several functions, we adopt the following notation which shows the dependence on both the point c and the function f:

$$Q_{c,f}(h) = \frac{f(c+h) - f(c)}{h}, \quad \text{where } h \neq 0.$$

Lemma F27

Let B be the blancmange function. Then for $m = 1, 2, \ldots,$ we have

$$B(x) = s(x) + \frac{1}{2}s(2x) + \cdots + \frac{1}{2^{m-1}}s(2^{m-1}x) + \frac{1}{2^m}B(2^m x),$$

and the function

$$x \longmapsto s(x) + \frac{1}{2}s(2x) + \cdots + \frac{1}{2^{m-1}}s(2^{m-1}x)$$

is linear on all intervals of the form $[p\,2^{-m}, (p+1)2^{-m}]$, where $p \in \mathbb{Z}$.

Proof Recall that a function is *linear* if it has a rule of the form $x \longmapsto ax + b$ for some $a, b \in \mathbb{R}$.

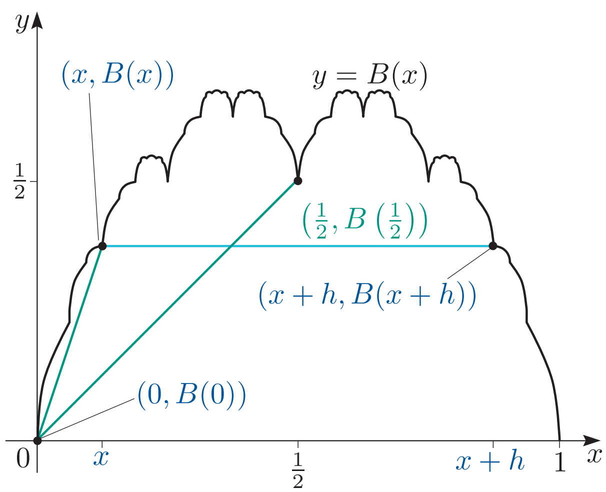
This follows directly from equation (2) and the definition of s. ∎

Lemma F28

Let B be the blancmange function. Then for each $x \in [0, 1]$, there exist $h, k \neq 0$ such that

$$x + h,\ x + k \in [0, 1] \quad \text{and} \quad |Q_{x,B}(h) - Q_{x,B}(k)| \geq 1.$$

Proof The proof is illustrated in the following figure.



Observe that the graph of $y = B(x)$ is symmetrical about the line $x = \frac{1}{2}$, and that $B(x) \geq x$ for $x \in \left[0, \frac{1}{2}\right]$.

We can assume by symmetry that $0 \le x \le \frac{1}{2}$. We now choose h so that $x + h = 1 - x$, and k so that $x + k = 0$ (or $k = \frac{1}{2}$ if $x = 0$). Then

$$Q_{x,B}(h) = \frac{B(x+h) - B(x)}{h} = \frac{B(1-x) - B(x)}{h} = 0,$$

by symmetry. Also, for $x \ne 0$,

$$Q_{x,B}(k) = \frac{B(x+k) - B(x)}{k} = \frac{B(0) - B(x)}{-x} \ge 1,$$

because $B(0) = 0$ and $B(x) \ge x$ for $x \in \left[0, \frac{1}{2}\right]$. A similar argument shows that the same result holds for $x = 0$. Thus the inequality in the statement of the lemma follows, as required. ■

Lemma F29

Given any real function f and a linear function $g(x) = ax + b$, the corresponding difference quotients of the functions f and $f + g$ always differ by a, the gradient of the linear function g.

Proof This holds because

$$Q_{c,f+g}(h) - Q_{c,f}(h) = Q_{c,g}(h) = (g(c+h) - g(c))/h = a,$$

for any $c \in \mathbb{R}$ and $h \ne 0$. ■

We now use these three lemmas to prove that the blancmange function is nowhere differentiable.

Proof of Theorem F26 Let $c \in \mathbb{R}$ and choose integers p_m, $m = 0, 1, 2, \ldots$, such that $c \in I_m$, where

$$I_m = \left[p_m 2^{-m}, \ (p_m + 1)2^{-m}\right].$$

Note that (I_m) is a 'nested' sequence of closed intervals, each of which contains the point c.

Now it follows from Lemma F27 that, on the interval I_m, the function B is the sum of a linear function and the function $B_m(x) = 2^{-m}B(2^m x)$, which is obtained from B by scaling in both the x- and y-directions by the factor 2^{-m}.

The graph of B_m on the interval I_m is a scaled-down copy of the graph of B on $[0, 1]$ (see Figure 8): it is a mini-blancmange!

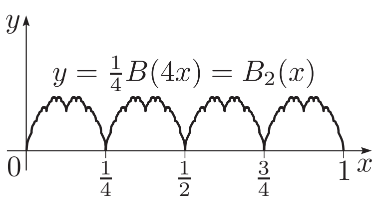


Figure 8 The graph of $B_2(x)$

Difference quotients are unchanged by such a scaling, so Lemma F28 implies that there exist $h_m, k_m \ne 0$ such that

$$c + h_m,\ c + k_m \in I_m \quad \text{and} \quad |Q_{c,B_m}(h_m) - Q_{c,B_m}(k_m)| \ge 1.$$

Therefore, by Lemma F29,

$$|Q_{c,B}(h_m) - Q_{c,B}(k_m)| = |Q_{c,B_m}(h_m) - Q_{c,B_m}(k_m)| \geq 1. \qquad (3)$$

This holds because the difference quotients of the linear part of the function B cancel out.

Now since c, $c + h_m$, $c + k_m \in I_m$, for $m = 1, 2, \dots$, and I_m has length 2^{-m}, we have $h_m \to 0$ and $k_m \to 0$. Thus if B is differentiable at c, then it follows from the definition of differentiability that

$$Q_{c,B}(h_m) \to B'(c) \quad \text{and} \quad Q_{c,B}(k_m) \to B'(c) \ \text{ as } m \to \infty.$$

But this contradicts inequality (3), so B is not differentiable at c. ■

Finally we note that, since the function B is continuous, it follows from the Extreme Value Theorem (see Subsection 3.3 of Unit D4) that it must have a maximum and a minimum on the interval $[0, 1]$. The minimum is $B(0) = B(1) = 0$, but the maximum is not so clear. In fact, it can be shown that the maximum is $B\left(\frac{1}{3}\right) = \frac{2}{3}$, and that this value is taken at infinitely many points in $[0, 1]$; a strange function indeed!

2 Rules for differentiation

In Section 1 you saw how to show that various basic functions are differentiable on $\mathbb{R}$, by working directly with the definition of a differentiable function. In this section you will see that we can often show that a function is differentiable by using the Combination Rules, the Composition Rule (also called the Chain Rule) and the Inverse Function Rule for differentiable functions.

We will use these rules to determine the derivatives of many more functions. The most important of these (together with the basic functions from Section 1) are summarised in a table of standard derivatives, which can be found at the end of this unit and in the module Handbook.

2.1 Combination Rules

The Combination Rules for differentiable functions are a consequence of the Combination Rules for limits that you met in Subsection 1.3 of Unit F1.

Theorem F30 Combination Rules for differentiable functions

Let f and g be defined on an open interval I, and let $c \in I$. If f and g are differentiable at c, then so are the following functions.

Sum Rule $f + g$, with derivative

$$(f+g)'(c) = f'(c) + g'(c)$$

Multiple Rule λf, for $\lambda \in \mathbb{R}$, with derivative

$$(\lambda f)'(c) = \lambda f'(c)$$

Product Rule fg, with derivative

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

Quotient Rule f/g, provided that $g(c) \neq 0$, with derivative

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{(g(c))^2}.$$

Each of these rules can be expressed in any of the alternative notations commonly used for derivatives. For example, in Leibniz notation, the Product Rule becomes

$$\text{if } y = uv, \text{ then } \frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}.$$

Similarly, the Quotient Rule is written

$$\text{if } y = u/v, \text{ then } \frac{dy}{dx} = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right).$$

Proof of Theorem F30

To prove each rule, we denote the function that we wish to show is differentiable by F, and express the difference quotient for F in terms of the difference quotients for f and g.

Suppose that f and g are both differentiable at c.

Sum Rule Let $F = f + g$. Then

$$\begin{aligned}\frac{F(x) - F(c)}{x - c} &= \frac{(f(x) + g(x)) - (f(c) + g(c))}{x - c}\\ &= \frac{f(x) - f(c)}{x - c} + \frac{g(x) - g(c)}{x - c}\\ &\to f'(c) + g'(c) \quad \text{as } x \to c,\end{aligned}$$

by the Sum Rule for limits, since f and g are differentiable at c. Thus F is differentiable at c, with derivative

$$F'(c) = f'(c) + g'(c).$$

Multiple Rule This is a special case of the Product Rule, with $g(x) = \lambda$.

Product Rule Let $F = fg$. Then

$$\begin{aligned}\frac{F(x)-F(c)}{x-c} &= \frac{f(x)g(x)-f(c)g(c)}{x-c}\\ &= \frac{f(x)-f(c)}{x-c}\,g(x) + f(c)\,\frac{g(x)-g(c)}{x-c}\\ &\to f'(c)g(c)+f(c)g'(c) \quad \text{as } x \to c,\end{aligned}$$

by the Combination Rules for limits, since f and g are differentiable at c, and g is continuous at c by Theorem F24, so that $g(x) \to g(c)$ as $x \to c$.

Thus F is differentiable at c, with derivative

$$F'(c) = f'(c)g(c) + f(c)g'(c).$$

Quotient Rule We first use the ε-δ definition of continuity that you met in Subsection 3.1 of Unit F1 to show that, since $g(c) \neq 0$ and g is continuous at c, then g must be non-zero on an open interval containing c. To show this, we take $\varepsilon = \frac{1}{2}|g(c)|$ in the definition.

Let $F = f/g$. Since g is continuous at c and $g(c) \neq 0$, there exists $\delta > 0$ such that $J = (c-\delta, c+\delta) \subseteq I$ and

$$|g(x) - g(c)| < \tfrac{1}{2}|g(c)|, \quad \text{for all } x \text{ with } |x-c| < \delta.$$

In particular, this shows that $g(x) \neq 0$ for $x \in J$, so the domain of F contains J. Then, for $x \in J$,

$$\begin{aligned}\frac{F(x)-F(c)}{x-c} &= \frac{1}{x-c}\left(\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}\right)\\ &= \frac{f(x)g(c)-f(c)g(x)}{(x-c)g(x)g(c)}\\ &= \frac{g(c)(f(x)-f(c)) - f(c)(g(x)-g(c))}{(x-c)g(x)g(c)}\\ &= \frac{1}{g(x)g(c)}\left(g(c)\,\frac{f(x)-f(c)}{x-c} - f(c)\,\frac{g(x)-g(c)}{x-c}\right)\\ &\to \frac{g(c)f'(c)-f(c)g'(c)}{(g(c))^2} \quad \text{as } x \to c,\end{aligned}$$

by the Combination Rules for limits, since f and g are differentiable at c, and g is continuous at c.

Thus F is differentiable at c, with derivative

$$F'(c) = \frac{g(c)f'(c)-f(c)g'(c)}{(g(c))^2}.$$

■

Since, by Theorem F21(b), the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$ for any $n \in \mathbb{N}$, it follows from the Combination Rules that any polynomial function is differentiable on $\mathbb{R}$ and that its derivative can be obtained by differentiating the polynomial term by term. We state this result as a corollary of the Combination Rules.

Corollary F31

Let

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \quad (x \in \mathbb{R}),$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Then p is differentiable on $\mathbb{R}$, with derivative

$$p'(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1} \quad (x \in \mathbb{R}).$$

Furthermore, since a rational function is a quotient of two polynomials, it follows from Corollary F31 and from the Quotient Rule that a rational function is differentiable at all points where its denominator is non-zero, that is, at all points of the domain of the function. Here is an example.

Worked Exercise F12

Prove that the function

$$f(x) = \frac{x^3}{x^2 - 1} \quad (x \in \mathbb{R} - \{-1, 1\})$$

is differentiable on its domain, and find its derivative.

Solution

The function f is a rational function of the form $f = p/q$, where $p(x) = x^3$ and $q(x) = x^2 - 1$, whose denominator q is non-zero on $\mathbb{R} - \{-1, 1\}$. Thus, by the Quotient Rule, f is differentiable on $\mathbb{R} - \{-1, 1\}$.

Here we write out all the steps in the application of the Quotient Rule, but it is not necessary for you to do this in your solutions.

Now $p'(x) = 3x^2$ and $q'(x) = 2x$. Thus, by the Quotient Rule, the derivative of f is

$$\begin{aligned} f'(x) &= \frac{(x^2 - 1)3x^2 - x^3(2x)}{(x^2 - 1)^2} \\ &= \frac{x^4 - 3x^2}{(x^2 - 1)^2} \quad (x \in \mathbb{R} - \{-1, 1\}). \end{aligned}$$

The Quotient Rule can also be used to show that the formula for differentiating $f(x) = x^n$, given in Theorem F21(b), remains valid if n is a *negative* integer.

We now give several exercises. It may not be necessary for you to work through all these exercises if you are confident that you can use the Combination Rules.

Exercise F17

Find the derivative of each of the following functions.

(a) $f(x) = x^7 - 2x^4 + 3x^3 - 5x + 1 \quad (x \in \mathbb{R})$

(b) $f(x) = \dfrac{x^2+1}{x^3-1} \quad (x \in \mathbb{R} - \{1\})$

(c) $f(x) = \sin x \cos x \quad (x \in \mathbb{R})$

(d) $f(x) = \dfrac{e^x}{3 + \sin x - 2\cos x} \quad (x \in \mathbb{R})$

Exercise F18

Find the third derivative of the function

$$f(x) = xe^x \quad (x \in \mathbb{R}).$$

In Section 1 we differentiated the functions sin, cos and exp. We now ask you to use these basic derivatives and the Combination Rules to find the derivatives of the other trigonometric functions and the three most common hyperbolic functions. The derivatives of these functions are included in the table of standard derivatives given at the end of this unit and in the module Handbook.

Exercise F19

Find the derivative of each of the following functions.

(a) $f(x) = \tan x \quad \left(x \neq (k + \tfrac{1}{2})\pi,\ k \in \mathbb{Z}\right)$

(b) $f(x) = \operatorname{cosec} x \quad (x \neq k\pi,\ k \in \mathbb{Z})$

(c) $f(x) = \sec x \quad \left(x \neq (k + \tfrac{1}{2})\pi,\ k \in \mathbb{Z}\right)$

(d) $f(x) = \cot x \quad (x \neq k\pi,\ k \in \mathbb{Z})$

Exercise F20

Find the derivative of each of the following functions.

(a) $f(x) = \sinh x \quad (x \in \mathbb{R})$

(b) $f(x) = \cosh x \quad (x \in \mathbb{R})$

(c) $f(x) = \tanh x \quad (x \in \mathbb{R})$

2.2 Composition Rule

In Subsection 2.1 we extended our stock of differentiable functions to include all polynomial, rational, trigonometric and hyperbolic functions. We also need to be able to differentiate functions such as

$$f(x) = \sin(\cos x) \quad (x \in \mathbb{R}),$$

which is the composite of the two differentiable functions sin and cos. To do this, we use the following Composition Rule for differentiable functions, which is commonly known as the Chain Rule.

Theorem F32 Composition Rule for differentiable functions (the Chain Rule)

Let f be defined on an open interval I, let g be defined on an open interval J such that $f(I) \subseteq J$, and let $c \in I$.

If f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Remarks

1. The Composition Rule tells us that 'a differentiable function of a differentiable function is differentiable', and gives us a formula for its derivative.
2. When written in Leibniz notation, the Composition Rule has a form that is easy to remember: if we put

   $$u = f(x) \quad \text{and} \quad y = g(u) = g(f(x)),$$

   then

   $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$
3. The Composition Rule can be extended to a composite of three or more functions; for example,

   $$(h \circ g \circ f)'(x) = h'(g(f(x)))g'(f(x))f'(x).$$

   In Leibniz notation, if we put

   $$v = f(x), \quad u = g(v) \quad \text{and} \quad y = h(u) = h(g(f(x))),$$

   then

   $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}.$$

   We often use this extended form of the Composition Rule without mentioning it explicitly.

Proof of Theorem F32 Let $F = g \circ f$. The difference quotient for F at c is

$$\frac{F(x) - F(c)}{x - c} = \frac{g(f(x)) - g(f(c))}{x - c}. \tag{4}$$

Let $y = f(x)$, where $x \in I$, and let $d = f(c)$. Then the right-hand side of equation (4) can be written

$$\left(\frac{g(y) - g(d)}{y - d}\right)\left(\frac{f(x) - f(c)}{x - c}\right), \quad \text{provided that } y \neq d. \tag{5}$$

To avoid the difficulty that expression (5) is undefined if $y = d$, which can occur in some situations, we introduce the function

$$h(y) = \begin{cases} \dfrac{g(y) - g(d)}{y - d}, & y \neq d, \\ g'(d), & y = d. \end{cases}$$

Since g is differentiable at d,

$$h(y) \to g'(d) \text{ as } y \to d;$$

and since $h(d) = g'(d)$, it follows that h is continuous at d.

By the Composition Rule for continuous functions (see Subsection 2.2 of Unit D4), and recalling that $y = f(x)$ and $d = f(c)$, we deduce that

$$(h \circ f)(x) = \begin{cases} \dfrac{g(f(x)) - g(f(c))}{f(x) - f(c)}, & f(x) \neq f(c), \\ g'(f(c)), & f(x) = f(c), \end{cases}$$

is continuous at c.

Next, note that if $f(x) \neq f(c)$, then equation (4) and expression (5) give

$$\frac{F(x) - F(c)}{x - c} = (h \circ f)(x)\left(\frac{f(x) - f(c)}{x - c}\right). \tag{6}$$

Equation (6) is also true when $f(x) = f(c)$, since both sides are then 0.

If we now let x tend to c in equation (6) and use the continuity at c of the function $h \circ f$, then we obtain

$$\frac{F(x) - F(c)}{x - c} \to g'(f(c))f'(c) \text{ as } x \to c.$$

Thus F is differentiable at c, with derivative

$$F'(c) = g'(f(c))f'(c).$$ ■

If you are interested in why we need to allow for the possibility that $y = d$ in the above proof, here is an example of a situation where this arises. Suppose that f is the function

$$f(x) = \begin{cases} x^2 \cos(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

which is differentiable at 0, as you showed in Exercise F15(b). Then if $c = 0$ we have $d = f(c) = 0$, and since $\cos(n + \frac{1}{2})\pi = 0$ for $n \in \mathbb{N}$, it follows that $y = f(x)$ takes the value 0 infinitely many times in any interval containing 0.

Worked Exercise F13

Prove that each of the following composite functions is differentiable on its domain, and find its derivative.

(a) $k(x) = \sin(\cos x) \quad (x \in \mathbb{R})$

(b) $k(x) = \cosh(e^{2x}) \quad (x \in \mathbb{R})$

(c) $k(x) = \tan(x^2) \quad (x \in (-1, 1))$

Solution

(a) Here $k(x) = \sin(\cos x)$, so let

$$f(x) = \cos x \quad \text{and} \quad g(x) = \sin x \quad (x \in \mathbb{R}).$$

Then f and g are differentiable on $\mathbb{R}$, and

$$f'(x) = -\sin x \quad \text{and} \quad g'(x) = \cos x \quad (x \in \mathbb{R}).$$

Here we write out all the steps in the application of the Composition Rule, but it is not necessary for you to do this in your solutions.

By the Composition Rule, $k = g \circ f$ is differentiable on $\mathbb{R}$, and

$$\begin{aligned} k'(x) &= g'(f(x))f'(x) \\ &= \cos(\cos x) \times (-\sin x) \\ &= -\cos(\cos x)\sin x. \end{aligned}$$

(b) Here $k(x) = \cosh(e^{2x})$, so let

$$f(x) = 2x, \quad g(x) = e^x \quad \text{and} \quad h(x) = \cosh x \quad (x \in \mathbb{R}).$$

Then f, g and h are differentiable on $\mathbb{R}$, and

$$f'(x) = 2, \quad g'(x) = e^x \quad \text{and} \quad h'(x) = \sinh x \quad (x \in \mathbb{R}).$$

Here we use the extended form of the Composition Rule.

By the Composition Rule, $k = h \circ g \circ f$ is differentiable on $\mathbb{R}$, and

$$\begin{aligned} k'(x) &= h'(g(f(x)))g'(f(x))f'(x) \\ &= \sinh(e^{2x}) \times e^{2x} \times 2 \\ &= 2e^{2x}\sinh(e^{2x}). \end{aligned}$$

(c) In the notation of the Composition Rule, we can write

$$f(x) = x^2 \quad (x \in I) \quad \text{and} \quad g(x) = \tan x,$$

where $I = (-1, 1)$. Then $f(I) = [0, 1)$, so if we choose $J = (-\pi/2, \pi/2)$, then $f(I) \subseteq J$, as required.

Now

$$f'(x) = 2x \quad (x \in (-1, 1))$$

and

$$g'(x) = \sec^2 x.$$

Thus, by the Composition Rule, $k = g \circ f$ is differentiable on $(-1, 1)$, and

$$\begin{aligned} k'(x) &= g'(f(x))f'(x) \\ &= 2x \sec^2(x^2) \quad (x \in (-1, 1)). \end{aligned}$$

Exercise F21

Find the derivative of each of the following functions.

(a) $f(x) = \sinh(x^2) \quad (x \in \mathbb{R})$

(b) $f(x) = \sin(\sinh 2x) \quad (x \in \mathbb{R})$

(c) $f(x) = \sin\left(\dfrac{\cos 2x}{x^2}\right) \quad (x \in (0, \infty))$

2.3 Inverse Function Rule

In Section 4 of Unit D4 we discussed inverse functions and proved that if a function f with domain an interval I and image set $J = f(I)$ is strictly monotonic and continuous on I, then J is an interval, and f possesses a strictly monotonic and continuous inverse function f^{-1} with domain J. (Recall that *strictly monotonic* means that f is either strictly increasing or strictly decreasing.) In particular, we showed that the power functions, the trigonometric functions, the exponential function and the hyperbolic functions all have continuous inverse functions, provided that we restrict their domains where necessary.

These standard functions are all differentiable on their domains, and we now investigate whether their inverse functions also have this property.

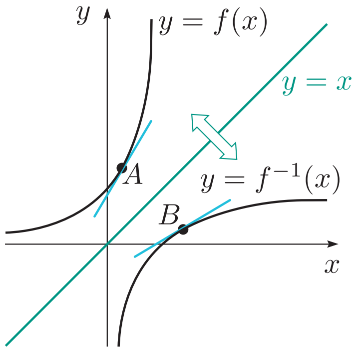


Figure 9 The tangent at a point on the graph $y = f(x)$ and its reflection in the line $y = x$

It is instructive to begin by considering their graphs. Recall that we obtain the graph $y = f^{-1}(x)$ by reflecting the graph $y = f(x)$ in the line $y = x$, which maps a typical point $A(c, d)$ on the graph $y = f(x)$ to the point $B(d, c)$ on the graph $y = f^{-1}(x)$, as illustrated in Figure 9. This suggests that if the gradient of the tangent to the graph $y = f(x)$ at the point A is $f'(c) = m$, then the gradient of the tangent to the graph $y = f^{-1}(x)$ at B is $(f^{-1})'(d) = 1/m$, provided that $m \neq 0$. However, if the graph of f has a horizontal tangent ($m = 0$) at a point A, then the graph of f^{-1} has a vertical tangent at the corresponding point B; in this case, f^{-1} is not differentiable at B, since $1/m$ is not defined for $m = 0$. We therefore need the condition '$f'(x)$ is non-zero' in our statement of the rule for differentiating inverse functions. The Inverse Function Rule for differentiable functions is formally stated as the next theorem and illustrated in Figures 10 and 11.

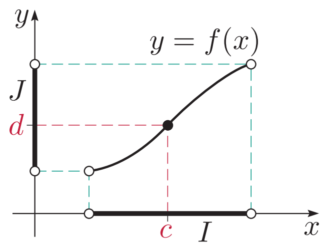


Figure 10 The graph of f in Theorem F33

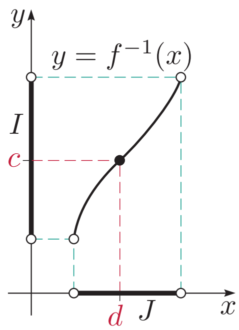


Figure 11 The graph of f^{-1} in Theorem F33

Theorem F33 Inverse Function Rule for differentiable functions

Let f be a function whose domain is an open interval I on which f is continuous and strictly monotonic. Then f has an inverse function f^{-1} with domain $J = f(I)$.

If f is differentiable on I and $f'(x) \neq 0$ for $x \in I$, then f^{-1} is differentiable on J. Also, if $c \in I$ and $d = f(c)$, then

$$(f^{-1})'(d) = \frac{1}{f'(c)}.$$

The Leibniz notation for derivatives can be used to express the Inverse Function Rule in a form that is easy to remember: if we put

$$y = f(x) \quad \text{and} \quad x = f^{-1}(y),$$

and write

$$\frac{dy}{dx} \text{ for } f'(x) \quad \text{and} \quad \frac{dx}{dy} \text{ for } (f^{-1})'(y),$$

then

$$\frac{dx}{dy} = \frac{1}{dy/dx}.$$

Proof of Theorem F33 First note that f has an inverse function f^{-1} with domain $J = f(I)$ by the Inverse Function Theorem for continuous functions, proved in Subsection 4.1 of Unit D4.

Let $y \in J$ and $y \neq d$, so $f^{-1}(y) = x$, where $x \in I$ and $x \neq c$ (since f is strictly monotonic).

Then the difference quotient for f^{-1} at d is

$$\begin{aligned}\frac{f^{-1}(y)-f^{-1}(d)}{y-d} &= \frac{x-c}{f(x)-f(c)}\\ &= 1\bigg/\frac{f(x)-f(c)}{x-c}.\end{aligned}$$

As $y \to d$, we have $x = f^{-1}(y) \to c$, since f^{-1} is continuous. So

$$\begin{aligned}\frac{f^{-1}(y)-f^{-1}(d)}{y-d} &= 1\bigg/\frac{f(x)-f(c)}{x-c}\\ &\to 1/f'(c) \quad \text{as } y \to d \quad (\text{since } f'(c) \neq 0).\end{aligned}$$

Thus f^{-1} is differentiable at d, with derivative $(f^{-1})'(d) = 1/f'(c)$. So f^{-1} is differentiable on J. ■

The next worked exercise shows how the Inverse Function Rule can be used to determine the derivative of the inverse for some standard functions. (The derivatives of these inverse functions are included in the table of standard derivatives given at the end of this unit and in the module Handbook.)

Worked Exercise F14

For each of the following functions f, state the domain and rule of f^{-1}, show that f^{-1} is differentiable and determine its derivative.

(a) $f(x) = x^n \quad (x \in \mathbb{R}^+)$, where $n \in \mathbb{N}$, $n \geq 2$

(b) $f(x) = \tan x \quad (x \in (-\pi/2, \pi/2))$

(c) $f(x) = e^x \quad (x \in \mathbb{R})$

Solution

(a) The function

$$f(x) = x^n \quad (x \in \mathbb{R}^+)$$

is continuous and strictly increasing, and $f((0,\infty)) = (0,\infty)$. Also, f is differentiable on $(0,\infty)$, and its derivative $f'(x) = nx^{n-1}$ is non-zero there. So f satisfies the conditions of the Inverse Function Rule.

Hence f has an inverse function f^{-1} and $f^{-1}(y) = y^{1/n}$ is differentiable on its domain $(0,\infty)$. If $y = f(x) = x^n$ (so that $x = y^{1/n}$), then

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{nx^{n-1}} = \frac{1}{n(y^{1/n})^{n-1}} = \frac{1}{n}\,y^{(1/n)-1}.$$

Replacing the domain variable y by x, we obtain

$$(f^{-1})'(x) = \frac{1}{n}x^{(1/n)-1} \quad (x \in (0,\infty)).$$

(b) The function

$$f(x) = \tan x \quad (x \in (-\pi/2, \pi/2))$$

is continuous and strictly increasing, and $f((-\pi/2, \pi/2)) = \mathbb{R}$. Also, f is differentiable on $(-\pi/2, \pi/2)$, and its derivative $f'(x) = \sec^2 x$ is non-zero there. So f satisfies the conditions of the Inverse Function Rule.

Hence f has an inverse function f^{-1} and $f^{-1}(y) = \tan^{-1} y$ is differentiable on its domain $\mathbb{R}$. If $y = f(x) = \tan x$, then

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{\sec^2 x} = \frac{1}{1 + \tan^2 x} = \frac{1}{1 + y^2}.$$

Replacing the domain variable y by x, we obtain

$$(\tan^{-1})'(x) = \frac{1}{1 + x^2} \quad (x \in \mathbb{R}).$$

(c) The function

$$f(x) = e^x \quad (x \in \mathbb{R})$$

is continuous and strictly increasing, and $f(\mathbb{R}) = (0, \infty)$. Also, f is differentiable on $\mathbb{R}$, and its derivative $f'(x) = e^x$ is non-zero there. So f satisfies the conditions of the Inverse Function Rule.

Hence f has an inverse function f^{-1} (which we call log) and $f^{-1}(y) = \log y$ is differentiable on its domain $(0, \infty)$. If $y = f(x) = e^x$, then

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{e^x} = \frac{1}{y}.$$

Replacing the domain variable y by x, we obtain

$$(\log)'(x) = \frac{1}{x} \quad (x \in (0, \infty)).$$

Exercise F22

For each of the following functions f, show that f^{-1} is differentiable and determine its derivative.

(a) $f(x) = \cos x \quad (x \in (0, \pi))$ (b) $f(x) = \sinh x \quad (x \in \mathbb{R})$

The Inverse Function Rule can sometimes be used to find values of the derivative of an inverse function f^{-1} even when the equation $y = f(x)$ cannot be solved to give a formula for the rule of f^{-1}. The next exercise illustrates this point.

Exercise F23

Let $f(x) = x^5 + x - 1$ $(x \in \mathbb{R})$.

(a) Prove that f has an inverse function f^{-1} which is differentiable on $\mathbb{R}$. (You may assume here that $f(\mathbb{R}) = \mathbb{R}$; this was proved in Worked Exercise D54 in Unit D4.)

(b) Find the values of $(f^{-1})'(d)$ at those points $d = f(c)$ where $c = 0, 1, -1$.

Exponential functions

In Subsection 4.3 of Unit D4 we defined the number a^x, for $a > 0$, by the formula

$$a^x = \exp(x \log a).$$

Since the functions exp and log are differentiable on $\mathbb{R}$ and $\mathbb{R}^+$, respectively (by Theorem F21(e) and Worked Exercise F14(c)), it follows that we can use this formula to determine the derivatives of several related functions. The functions in the next two worked exercises are included in the table of standard derivatives. Notice that the derivative in Worked Exercise F15 agrees with the formula for the derivative of $f(x) = x^n$, where $n \in \mathbb{N}$.

Worked Exercise F15

Prove that, for $\alpha \in \mathbb{R}$, the function

$$f(x) = x^\alpha \quad (x \in \mathbb{R}^+)$$

is differentiable on its domain, and that

$$f'(x) = \alpha x^{\alpha - 1} \quad (x \in \mathbb{R}^+).$$

Solution

By definition,

$$f(x) = \exp(\alpha \log x) \quad (x \in \mathbb{R}^+).$$

The function $x \longmapsto \alpha \log x$ is differentiable on $\mathbb{R}^+$, with derivative α/x. Thus, by the Composition Rule, f is differentiable on $\mathbb{R}^+$, with derivative

$$\begin{aligned} f'(x) &= \exp(\alpha \log x) \times (\alpha/x) \\ &= x^\alpha(\alpha/x) = \alpha x^{\alpha-1} \quad (x \in \mathbb{R}^+). \end{aligned}$$

Worked Exercise F16

Prove that, for $a > 0$, the function

$$f(x) = a^x \quad (x \in \mathbb{R})$$

is differentiable, and that

$$f'(x) = a^x \log a \quad (x \in \mathbb{R}).$$

Solution

By definition,

$$f(x) = \exp(x \log a) \quad (x \in \mathbb{R}).$$

The function $x \longmapsto x \log a$ is differentiable on $\mathbb{R}$, with derivative $\log a$. By the Composition Rule, f is differentiable on $\mathbb{R}$, with derivative

$$\begin{aligned} f'(x) &= \exp(x \log a) \times \log a \\ &= a^x \log a. \end{aligned}$$

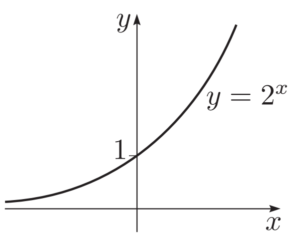


Figure 12 The graph of $f(x) = 2^x$

At the beginning of Book D we posed the following question: does the graph of the function $f(x) = 2^x$, shown in Figure 12, have a jump at the point $\sqrt{2}$? We showed in Subsection 4.3 of Unit D4 that f is continuous, which proves that its graph has no jumps. In Worked Exercise F16 we have now shown that f also is differentiable, so its graph has a tangent at every point and thus has no corners.

Exercise F24

Prove that the function

$$f(x) = x^x \quad (x \in \mathbb{R}^+)$$

is differentiable, and find its derivative.

3 Rolle's Theorem

In this section and in Section 4 you will meet some of the fundamental properties of functions that are differentiable not just at a particular point, but *on an interval*. These results are motivated by the geometric significance of differentiability in terms of tangents, and they explain why the graphs of differentiable functions possess certain geometric properties.

3.1 Local Extreme Value Theorem

In Section 3 of Unit D4 you met some of the fundamental properties of functions which are continuous on a bounded closed interval. In particular, you studied the Extreme Value Theorem, which states that if a function f is continuous on a closed interval $[a, b]$, then there are points c and d in $[a, b]$ such that

$$f(x) \leq f(d), \quad \text{for } x \in [a, b],$$

and

$$f(x) \geq f(c), \quad \text{for } x \in [a, b].$$

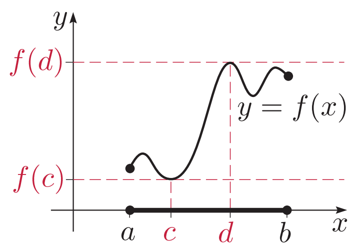


Figure 13 The extreme values of a function continuous on a closed interval

This is illustrated in Figure 13. The value $f(d)$ is the **maximum** of f on $[a, b]$, and the value $f(c)$ is the **minimum** of f on $[a, b]$. A maximum or a minimum of f is called an **extreme value** of f. But how do we determine the points c and d where these extreme values occur? In general, this is not easy. However, if the function f is differentiable, then we can, in principle, determine c and d by first finding any *local* extreme values of the function f on the interval $[a, b]$.

Roughly speaking, for a point c in (a, b), the value $f(c)$ is a *local maximum* of f on $[a, b]$ if $f(c)$ is the greatest value of f in the immediate vicinity of c, and a *local minimum* of f on $[a, b]$ if $f(c)$ is the least value of f in the immediate vicinity of c. These ideas are illustrated in Figure 14 and stated formally in the definitions which follow.

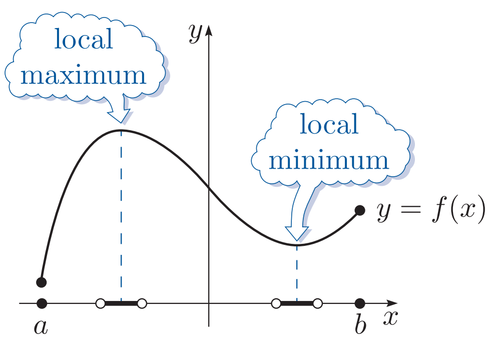


Figure 14 The local extreme values of a function

Definitions

The function f has

- a **local maximum** $f(c)$ at c if there is an open interval $I = (c-r, c+r)$, where $r > 0$, in the domain of f such that
$$f(x) \leq f(c), \quad \text{for } x \in I$$
- a **local minimum** $f(c)$ at c if there is an open interval $I = (c-r, c+r)$, where $r > 0$, in the domain of f such that
$$f(x) \geq f(c), \quad \text{for } x \in I$$
- a **local extreme value** $f(c)$ at c if $f(c)$ is either a local maximum or a local minimum.

By definition, a local extreme value of a function f defined on a bounded closed interval $[a, b]$ is an interior point of $[a, b]$; that is, a local extreme value cannot occur at either of the endpoints a and b.

If we want to find the local extreme values of a differentiable function f, then we can use the following result, which gives a connection between the local extreme values of a function f and the points c where $f'(c) = 0$. A point c such that $f'(c) = 0$ is called a **stationary point** of f; we sometimes say that f' **vanishes** at c.

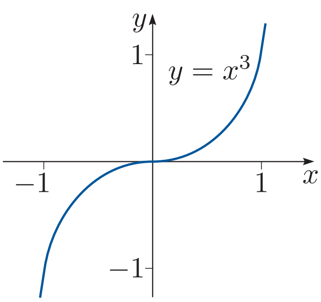


Figure 15 The graph of $f(x) = x^3$

Theorem F34 Local Extreme Value Theorem

If f has a local extreme value at c and f is differentiable at c, then
$$f'(c) = 0.$$

Note that the converse of the Local Extreme Value Theorem is *false*: a point where the derivative vanishes is not necessarily a local extreme value. For example, the function $f(x) = x^3$ does not have a local extreme value at 0, although $f'(0) = 0$; see Figure 15.

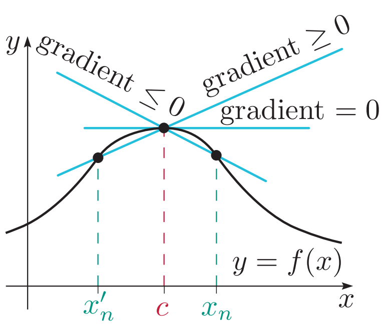


Figure 16 The proof of the Local Extreme Value Theorem

Proof of Theorem F34 We prove the result only for a local maximum; the proof of the local minimum version is similar. Suppose that f has a local maximum at c. Then there exists a positive number r such that
$$f(x) \leq f(c), \quad \text{for } c - r < x < c + r. \tag{7}$$

Now let
$$x_n = c + \frac{r}{n} \quad \text{and} \quad x'_n = c - \frac{r}{n}, \quad n = 2, 3, \ldots.$$

We do not include $n = 1$ as we require x_n and x'_n to lie in the open interval $(c - r, c + r)$. The argument which follows is illustrated in Figure 16.

Then $c < x_n < c + r$, for $n = 2, 3, \ldots$, so

$$f(x_n) - f(c) \leq 0 \quad \text{and} \quad x_n - c > 0, \quad \text{for } n = 2, 3, \ldots,$$

by inequality (7). Hence

$$\frac{f(x_n) - f(c)}{x_n - c} \leq 0, \quad \text{for } n = 2, 3, \ldots.$$

Since $x_n \to c$, we deduce by the Limit Inequality Rule (see Subsection 3.3 of Unit D2 *Sequences*) that

$$f'(c) = \lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} \leq 0. \tag{8}$$

On the other hand, $c - r < x'_n < c$, for $n = 2, 3, \ldots$, so

$$f(x'_n) - f(c) \leq 0 \quad \text{and} \quad x'_n - c < 0, \quad \text{for } n = 2, 3, \ldots,$$

by inequality (7). Hence

$$\frac{f(x'_n) - f(c)}{x'_n - c} \geq 0, \quad \text{for } n = 2, 3, \ldots.$$

Since $x'_n \to c$, we deduce that

$$f'(c) = \lim_{n\to\infty} \frac{f(x'_n) - f(c)}{x'_n - c} \geq 0. \tag{9}$$

Hence, by inequalities (8) and (9), we have $f'(c) = 0$, as required. ■

Any extreme value of a function f on a bounded closed interval $[a, b]$ which is not $f(a)$ or $f(b)$ must also be a local extreme value. Thus, by Theorem F34, such a point x must satisfy $f'(x) = 0$. This gives the following property of the extreme values of a differentiable function on a bounded closed interval.

Corollary F35

Let f be continuous on the closed interval $[a, b]$ and differentiable on (a, b). Then the extreme values of f on $[a, b]$ can occur only at a or b, or at points x in (a, b) where $f'(x) = 0$.

We now reformulate Corollary F35 as a strategy for locating maxima and minima.

Strategy F6

To find the maximum and minimum of a function f that is continuous on $[a, b]$ and differentiable on (a, b), do the following.

1. Determine the points $c_1, c_2, \ldots$ in (a, b) where f' is zero.
2. Hence determine the values of

   $$f(a),\ f(b),\ f(c_1),\ f(c_2),\ \ldots;$$

   the greatest of these is the maximum and the least is the minimum.

Note that, in some cases, there may be infinitely many points in (a, b) where f' is zero; this is so, for example, if f is constant on (a, b).

Exercise F25

Use Strategy F6 to determine the maximum and minimum of the function

$$f(x) = \sin^2 x + \cos x$$

on the interval $[0, \pi/2]$.

3.2 Rolle's Theorem

In the last subsection you saw that if a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then the extreme values of f can occur only at a or b, or at some point c in (a, b) such that $f'(c) = 0$. If we also know that the values of $f(a)$ and $f(b)$ are equal, then we can deduce that there must be some point c in (a, b) where $f'(c) = 0$. This is illustrated in Figure 17.

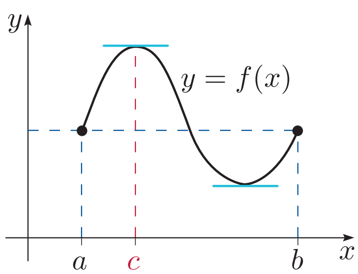


Figure 17 A function with $f(a) = f(b)$

Theorem F36 Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point c, with $a < c < b$, such that

$$f'(c) = 0.$$

Remarks

1. Rolle's Theorem is one of the most important theorems in analysis; for example, most of the results you will meet in Sections 4 and 5 depend on Rolle's Theorem.
2. Rolle's Theorem is an *existence theorem*; that is, it tells us that a point c with the stated property exists, but not how to find it. Often it is difficult to evaluate c explicitly, and there may be more than one point c in (a, b) at which f' vanishes.
3. In geometric terms, Rolle's Theorem states that if the line joining the points $(a, f(a))$, $(b, f(b))$ on the graph of f is horizontal, then so is the tangent to the graph for some $c \in (a, b)$.

Rolle's Theorem is one of the foundational theorems in differential calculus. Its importance lies in the fact that it is needed in the proof of the Mean Value Theorem and for establishing the existence of Taylor series. However, when Michel Rolle (1652–1719) made the first statement of the theorem in 1690, Taylor series had not yet been discovered and calculus was still in its infancy. Moreover, Rolle was deeply suspicious of its methods. The first appearance of his theorem was not in the context of calculus at all, but in the context of solving equations. It is paradoxical that the name of a man renowned for his opposition to the infinitesimal calculus should end up attached to one of the fundamental theorems in the subject.

During the eighteenth century the theorem lived on as a theorem in algebra until it appeared in the work of Leonhard Euler (1707–1783) in 1755. Euler, with the calculus at his fingertips, had no need for Rolle's rather convoluted algebraic method and so for the first time the theorem resembled its modern counterpart. It is not known whether Euler had read Rolle's work since, characteristically, he made no reference to any earlier work. By the 1830s the theorem had appeared in a number of textbooks on the theory of equations, but was still not associated with Rolle. It was first ascribed to Rolle by Wilhelm Drobisch (1802–1896), a professor at the University of Leipzig, in a textbook of 1834.

In the latter half of the nineteenth century the theorem underwent its second significant change. From being a useful result in the theory of equations it was transformed into a fundamental theorem in analysis. In 1873 Charles Hermite (1822–1901) used the theorem in his *Cours d'Analyse* in the context of the theory of Taylor series, clearly attributing it to Rolle. Hermite was the leading French analyst of his generation and his *Cours d'Analyse* was extremely influential in France during the latter part of the nineteenth century. His unequivocal association of the theorem with Rolle was decisive for future writers.

Charles Hermite

Proof of Theorem F36 Suppose that $f(a) = f(b)$.

If f is constant on $[a, b]$, then $f'(x) = 0$ everywhere on (a, b); in this case, we can take c to be any point of (a, b).

If f is non-constant on $[a, b]$, then either the maximum or the minimum (or both) of f on $[a, b]$ is different from the common value $f(a) = f(b)$.

Since f is continuous on $[a, b]$, it must have both a maximum and a minimum on $[a, b]$, by the Extreme Value Theorem; see Subsection 3.3 of Unit D4.

Since one of the extreme values occurs at some point c with $a < c < b$, the Local Extreme Value Theorem shows that $f'(c)$ must be zero. ■

We can use Rolle's Theorem to verify the existence of zeros of certain functions which are derivatives.

Worked Exercise F17

Use Rolle's Theorem to show that if

$$f(x) = 3x^4 - 2x^3 - 2x^2 + 2x,$$

then there is a value of c in $(-1, 1)$ such that $f'(c) = 0$.

Solution

Since f is a polynomial function, f is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Also, $f(1) = f(-1) = 1$. Thus f satisfies the conditions of Rolle's Theorem on $[-1, 1]$. It follows that there exists a point $c \in (-1, 1)$ such that $f'(c) = 0$.

For the function in Worked Exercise F17 we can find a value for c directly by using the fact that

$$\begin{aligned} f'(x) &= 12x^3 - 6x^2 - 4x + 2 \\ &= 2(3x^2 - 1)(2x - 1). \end{aligned}$$

Thus f' has a zero at each of the points $-1/\sqrt{3}$, $1/\sqrt{3}$ and $\frac{1}{2}$, which are all in $(-1, 1)$.

Exercise F26

Use Rolle's Theorem to show that if

$$f(x) = x^4 - 4x^3 + 3x^2 + 2,$$

then there is a value of c in $(1, 3)$ such that $f'(c) = 0$.

Exercise F27

For each of the following functions, state whether Rolle's Theorem applies for the given interval.

(a) $f(x) = \tan x, \quad [0, \pi]$

(b) $f(x) = 3|x - 1| - x, \quad [0, 3]$

(c) $f(x) = x - 9x^{17} + 8x^{18}, \quad [0, 1]$

(d) $f(x) = \sin x + \tan^{-1} x, \quad [0, \pi/2]$

4 Mean Value Theorem

In this section you will continue to study the geometric properties of functions that are differentiable on intervals and meet some of their applications.

4.1 Mean Value Theorem

First we recall the geometric interpretation of Rolle's Theorem from the previous section. Rolle's Theorem tells us that, under suitable conditions, if the chord joining the points $(a, f(a))$ and $(b, f(b))$ on the graph of f is horizontal, then so is the tangent to the graph for some c in (a, b).

If you imagine pushing this horizontal chord, always parallel to its original position, until it is just about to lose contact with the graph of f, then it appears that at this point the chord is a tangent to the graph; see the top graph in Figure 18. This 'chord-pushing' approach suggests that even if the original chord is not horizontal (that is, if $f(a) \neq f(b)$), then there must still be a point c in (a, b) at which the tangent is parallel to the chord; see the bottom graph in Figure 18.

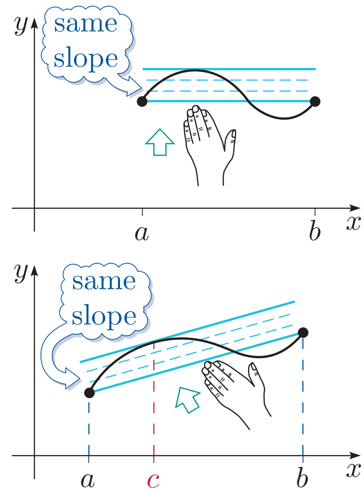


Figure 18 A tangent parallel to a chord

Worked Exercise F18

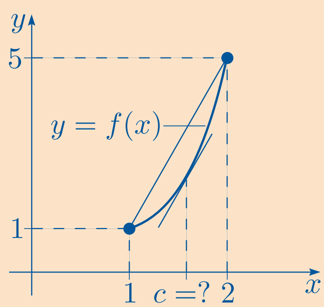
Consider the function

$$f(x) = x^3 - 3x + 3 \quad (x \in [1, 2]).$$

Find a point $c \in (1, 2)$ such that the tangent to the graph of f is parallel to the chord joining $(1, f(1))$ to $(2, f(2))$.

Solution

The graph of f is shown below.



Since $f(1) = 1 - 3 + 3 = 1$ and $f(2) = 8 - 6 + 3 = 5$, the gradient of the chord is

$$\frac{f(2) - f(1)}{2 - 1} = \frac{5 - 1}{2 - 1} = 4.$$

Now f is a polynomial, so it is differentiable on $(1,2)$, and its derivative is $f'(x) = 3x^2 - 3$. Hence $f'(c) = 4$ when $3c^2 = 7$; that is, when $c = \sqrt{7/3} \simeq 1.53$.

Thus at the point $(c, f(c))$ the tangent to the graph is parallel to the chord joining the endpoints of the graph.

We now generalise Rolle's Theorem and show that there is always a point where the tangent to the graph is parallel to the chord joining the endpoints. This result is known as the *Mean Value Theorem*, so-called since the gradient of the chord,

$$\frac{f(b) - f(a)}{b - a},$$

can be thought of as the *mean value* of the derivative between a and b.

Theorem F37 Mean Value Theorem

Let f be continuous on the closed interval $[a,b]$ and differentiable on (a,b). Then there exists a point c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remarks

1. Like Rolle's Theorem, this result is an *existence theorem*: it does not tell us what the point c is, just that such a point exists.
2. When $f(a) = f(b)$, the Mean Value Theorem reduces to Rolle's Theorem.

Proof of Theorem F37 The gradient of the chord joining the points $(a, f(a))$ and $(b, f(b))$ is

$$m = \frac{f(b) - f(a)}{b - a},$$

so the equation of the chord is

$$y = m(x - a) + f(a).$$

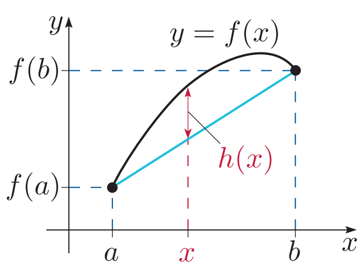


Figure 19 The vertical distance $h(x)$

It follows that, for $x \in [a,b]$, the vertical distance $h(x)$ from the point $(x, f(x))$ to the chord, as shown in Figure 19, is given by the function

$$h(x) = f(x) - (m(x - a) + f(a)).$$

Now $h(a) = h(b) = 0$, and h is continuous on $[a,b]$ and differentiable on (a,b). Thus h satisfies all the conditions of Rolle's Theorem.

It follows that there exists a point c in (a, b) such that $h'(c) = 0$. Since $h'(c) = f'(c) - m$, we have

$$f'(c) = m = \frac{f(b) - f(a)}{b - a},$$

as required. ■

Worked Exercise F19

Use the Mean Value Theorem to show that if

$$f(x) = \frac{x - 1}{x + 1},$$

then there is a point c in $\left(1, \frac{7}{2}\right)$ such that $f'(c) = \frac{2}{9}$.

Solution

The function f is a rational function whose denominator is non-zero on $\left[1, \frac{7}{2}\right]$, so f is continuous on $\left[1, \frac{7}{2}\right]$ and differentiable on $\left(1, \frac{7}{2}\right)$. Thus f satisfies the conditions of the Mean Value Theorem.

Now

$$\frac{f\left(\frac{7}{2}\right) - f(1)}{\frac{7}{2} - 1} = \frac{\frac{5}{9} - 0}{\frac{5}{2}} = \frac{2}{9}.$$

Thus, by the Mean Value Theorem, there exists a point c in $\left(1, \frac{7}{2}\right)$ such that $f'(c) = \frac{2}{9}$.

For the function in Worked Exercise F19 we can find a value for c directly. Since $f'(x) = 2/(x + 1)^2$, the point c satisfies

$$f'(c) = \frac{2}{(c + 1)^2} = \frac{2}{9}; \quad \text{that is,} \quad (c + 1)^2 = 9.$$

This equation has solutions 2 and -4, so $c = 2$ $\left(\text{since } 2 \in \left(1, \frac{7}{2}\right)\right)$.

In the following exercise, a value for c cannot be found in this direct way.

Exercise F28

Use the Mean Value Theorem to show that if

$$f(x) = xe^x,$$

then there is a point c in $(0, 2)$ such that $f'(c) = e^2$.

4.2 Positive, negative and zero derivatives

We now study some consequences of the Mean Value Theorem for functions whose derivatives are always positive, always negative, or always zero. First we prove a fundamental result about monotonic functions, which you used to help with graph sketching in Section 2 of Unit A4 *Real functions, graphs and conics.*

For any interval I, the **interior** of I is the largest open subinterval of I. It is obtained from I by removing any endpoints of I, so it consists of all the interior points of I.

Theorem F38 Increasing–Decreasing Theorem

Let f be continuous on an interval I and differentiable on the interior of I.

(a) If $f'(x) \geq 0$ for x in the interior of I, then f is increasing on I.

(b) If $f'(x) \leq 0$ for x in the interior of I, then f is decreasing on I.

Proof Choose any two points x_1 and x_2 in I, with $x_1 < x_2$. The function f satisfies the conditions of the Mean Value Theorem on the interval $[x_1, x_2]$, so there exists a point c in (x_1, x_2) such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c).$$

Hence $f(x_2) - f(x_1)$ has the same sign as $f'(c)$, so

(a) if $f'(x) \geq 0$ for x in the interior of I, then $f(x_2) - f(x_1) \geq 0$ and hence f is increasing on I

(b) if $f'(x) \leq 0$ for x in the interior of I, then $f(x_2) - f(x_1) \leq 0$ and hence f is decreasing on I. ■

There is also a version of Theorem F38 with the weak inequalities replaced by strict inequalities, in which case the conclusions are as follows.

(a) If $f'(x) > 0$ for x in the interior of I, then f is strictly increasing on I.

(b) If $f'(x) < 0$ for x in the interior of I, then f is strictly decreasing on I.

Note that the converse of each of these two statements with strict inequalities is *false*. For example, $f(x) = x^3$ is strictly increasing on $[-1, 1]$, but $f'(0) = 0$.

Exercise F29

On the given interval I, determine whether each of the following functions f is

- strictly increasing
- increasing, but not strictly increasing
- strictly decreasing, or
- decreasing, but not strictly decreasing.

(a) $f(x) = 3x^{4/3} - 4x, \quad I = [1, \infty)$

(b) $f(x) = x - \log x, \quad I = (0, 1]$

The following corollary to Theorem F38 will be useful in later units.

Corollary F39 Zero Derivative Theorem

Let f be continuous on an interval I and differentiable on the interior of I. If $f'(x) = 0$ for all x in the interior of I, then f is constant on I.

Proof Theorem F38(a) and (b) both apply, so f is both increasing and decreasing on I. Hence f is constant on I. ■

We can often determine whether a point c such that $f'(c) = 0$ is a local maximum or a local minimum of a function f by using the following test.

Theorem F40 Second Derivative Test

Let f be a twice-differentiable function defined on an open interval I containing a point c such that $f'(c) = 0$ and f'' is continuous at c.

(a) If $f''(c) > 0$, then $f(c)$ is a local minimum of f.

(b) If $f''(c) < 0$, then $f(c)$ is a local maximum of f.

Proof We prove part (a); the proof of part (b) is similar, so we omit it.

We first use the ε-δ definition of continuity with $\varepsilon = \frac{1}{2}f''(c)$ to show that $f''(x) > 0$ for x close to c. A similar technique was used in the proof of the Quotient Rule in Subsection 2.1.

Suppose that $f''(c) > 0$. Since f'' is continuous at c, there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq I$ and

$$|f''(x) - f''(c)| < \tfrac{1}{2}f''(c), \quad \text{for } x \in (c - \delta, c + \delta),$$

so

$$f''(x) > \tfrac{1}{2}f''(c) > 0, \quad \text{for } x \in (c - \delta, c + \delta).$$

Thus f' is strictly increasing on the open interval $(c - \delta, c + \delta)$, by the strict inequalities version of the Increasing–Decreasing Theorem. Since $f'(c) = 0$, we deduce that

$$f'(x) < 0, \quad \text{for } x \in (c - \delta, c),$$
$$f'(x) > 0, \quad \text{for } x \in (c, c + \delta).$$

Thus f has a local minimum at c, by another application of the strict inequalities version of the Increasing–Decreasing Theorem. ■

Note that if $f''(c) = 0$, then the Second Derivative Test gives us no information about local extreme values. For example, the function $f(x) = x^3$ satisfies $f'(0) = 0$ and $f''(0) = 0$, but it has neither a local maximum nor a local minimum at 0.

Exercise F30

Consider the function

$$f(x) = x^3 - 3x^2 + 1.$$

(a) Determine those points c such that $f'(c) = 0$.

(b) Using the Second Derivative Test, determine whether the points c found in part (a) correspond to local maxima or local minima, and find the values of these local maxima or local minima.

Proving inequalities

We now demonstrate how the Increasing–Decreasing Theorem can be used to prove certain inequalities involving differentiable functions.

First we prove a generalisation of Bernoulli's Inequality. Recall from Subsection 3.5 of Unit D1 that Bernoulli's Inequality states that

$$(1 + x)^n \geq 1 + nx, \quad \text{for } x \geq -1 \text{ and } n \in \mathbb{N}.$$

In the next worked exercise we show that this inequality still holds if we replace n by any real number $\alpha \geq 1$.

Worked Exercise F20

Let $\alpha \geq 1$. Prove that

$$(1+x)^\alpha \geq 1 + \alpha x, \quad \text{for } x \geq -1.$$

Solution

The case $\alpha = 1$ holds by Bernoulli's Inequality, so we can assume that $\alpha > 1$. Define the function

$$f(x) = (1+x)^\alpha - (1+\alpha x) \quad (x \in [-1, \infty)).$$

We want to show that $f(x) \geq 0$ for $x \in [-1, \infty)$. Since $f(0) = 1 - 1 = 0$, this is equivalent to showing that

$$f(x) \geq f(0), \quad \text{for } x \in [-1, \infty).$$

We do this by showing that

$$f \text{ is increasing on } (0, \infty) \text{ and decreasing on } [-1, 0). \qquad (*1)$$

Now the function f is continuous on $[-1, \infty)$ and differentiable on $(-1, \infty)$, with derivative

$$\begin{aligned} f'(x) &= \alpha(1+x)^{\alpha-1} - \alpha \\ &= \alpha((1+x)^{\alpha-1} - 1), \quad \text{for } x \in (-1, \infty). \qquad (*2) \end{aligned}$$

If $x > 0$, then $1 + x > 1$, so

$$(1+x)^{\alpha-1} > 1, \quad \text{for } x > 0.$$

Here we have used Rule 5 for rearranging inequalities from Section 2 of Unit D1:

$$\text{if } a, b \geq 0 \text{ and } p > 0, \text{ then } a > b \iff a^p > b^p.$$

We have used this with $p = \alpha - 1 > 0$.

Hence, by equation $(*2)$,

$$f'(x) > 0, \quad \text{for } x > 0,$$

so f is increasing on $(0, \infty)$, by the Increasing–Decreasing Theorem.

Similarly, if $-1 < x < 0$, then $0 < 1 + x < 1$, so

$$(1+x)^{\alpha-1} < 1, \quad \text{for } -1 < x < 0.$$

Hence, by equation $(*2)$,

$$f'(x) < 0, \quad \text{for } -1 < x < 0,$$

so f is decreasing on $(-1, 0)$. This proves statement $(*1)$.

Worked Exercise F20 illustrates the following general strategy for using the Increasing–Decreasing Theorem to prove inequalities.

Strategy F7

To prove that $g(x) \geq h(x)$, for $x \in [a, b]$, carry out the following steps.

1. Let
$$f(x) = g(x) - h(x),$$
and show that f is continuous on $[a, b]$ and differentiable on (a, b).
2. Prove that
either $f(a) \geq 0$ and $f'(x) \geq 0$ for $x \in (a, b)$,
or $f(b) \geq 0$ and $f'(x) \leq 0$ for $x \in (a, b)$.

The diagrams in Figure 20 below illustrate why Strategy F7 works.

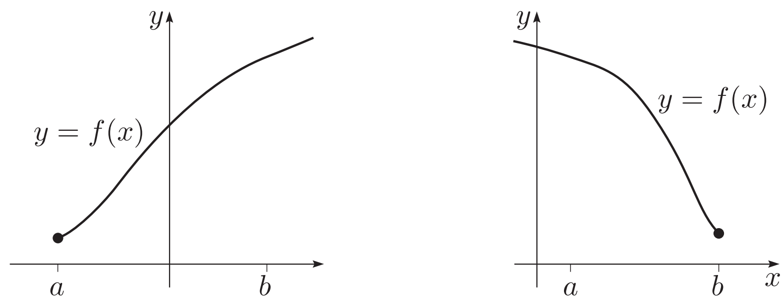


Figure 20 The two cases in Strategy F7

Remarks

1. There is a corresponding version of Strategy F7 in which the weak inequalities are replaced by strict inequalities.
2. We can also apply Strategy F7 to intervals of the form $[a, \infty)$ if the first case in step 2 holds, and to intervals of the form $(-\infty, b]$ if the second case in step 2 holds.
3. Notice that in Worked Exercise F20 we used *both* cases in step 2 of Strategy F7.

Exercise F31

Prove the following inequalities.

(a) $\sin x \leq x$, for $x \in [0, \infty)$ (b) $\frac{2}{3}x + \frac{1}{3} \geq x^{2/3}$, for $x \in [0, 1]$

5 L'Hôpital's Rule

In Section 1 we found the derivatives of sin and exp by using the results

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0}\frac{e^x - 1}{x} = 1,$$

which you met in Section 1 of Unit F1. Each of the above limits is of the form

$$\lim_{x\to c}\frac{f(x)}{g(x)},$$

where f and g are continuous functions with $f(c) = g(c) = 0$. Such limits cannot be evaluated by the Quotient Rule for limits, because this rule requires $\lim_{x\to c} g(x) \neq 0$.

There are similar problems with the following more complicated limits:

$$\lim_{x\to\pi/2}\frac{\cos 3x}{\sin x - e^{\cos x}} \quad \text{and} \quad \lim_{x\to 0}\frac{x^2}{\cosh x - 1}.$$

Do they exist? And if they do, what are their values?

In this section you will meet a result called l'Hôpital's Rule, which enables us to answer such questions.

5.1 Cauchy's Mean Value Theorem

In Section 4 you met the Mean Value Theorem which asserts that, under certain conditions, a function defined on a closed interval has the property that at some intermediate point, the tangent to its graph is parallel to the chord joining the endpoints. To prove l'Hôpital's Rule, we will need the following generalisation of the Mean Value Theorem.

Theorem F41 Cauchy's Mean Value Theorem

Let f and g be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $c \in (a,b)$ such that

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a));$$

in particular, if $g(b) \neq g(a)$ and $g'(c) \neq 0$, then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Remarks

1. Note that Cauchy's Mean Value Theorem involves *two* functions defined on a closed interval $[a, b]$ and, subject to the stated conditions, gives us an expression for the ratio of their derivatives at some point $c \in (a, b)$. (It is this expression that we need for the proof of l'Hôpital's Rule.) If we put $g(x) = x$, then Cauchy's Mean Value Theorem reduces to the usual Mean Value Theorem.

2. There is a geometric interpretation of Cauchy's Mean Value Theorem that you may find helpful. Recall from Subsection 5.4 of Unit A4 that we can describe a curve in $\mathbb{R}^2$ by specifying the x- and y-coordinates of its points using two functions, f and g, to define parametric equations

   $$x = g(t), \quad y = f(t),$$

   where the parameter t belongs to some suitable interval $[a, b]$. Thus we can think of the two functions in Cauchy's Mean Value Theorem as defining a curve in this way; see Figure 21.

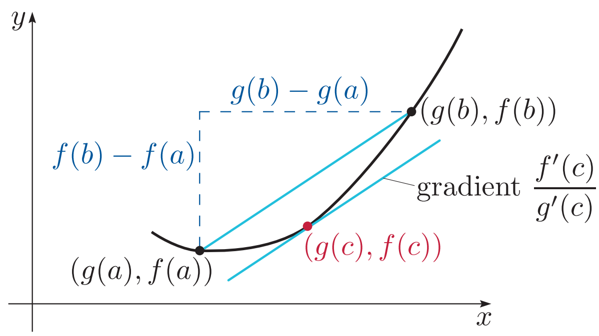
   

   Figure 21 A geometric interpretation of Cauchy's Mean Value Theorem

   Now it can be shown that, subject to certain conditions, the gradient of the tangent to this curve at the point $(g(c), f(c))$ is $\dfrac{f'(c)}{g'(c)}$ (though we do not prove this here). Thus, Cauchy's Mean Value Theorem tells us that there is some value c in the interval (a, b) such that the gradient of the curve at the point $(g(c), f(c))$ is equal to the gradient $\dfrac{f(b) - f(a)}{g(b) - g(a)}$ of the chord joining the endpoints of the curve, $(g(a), f(a))$ and $(g(b), f(b))$.

From 1815 to 1830 Augustin-Louis Cauchy (1789–1857) taught at the famous École Polytechnique in Paris, the École founded for the training of engineers. Cauchy wrote several textbooks on analysis designed for the students there, including his *Cours d'Analyse* of 1821 and his *Résumé des leçons données a l'École Royale Polytechnique sur le calcul infinitésimal* of 1823, the latter containing the result now known as the Cauchy (or Generalised) Mean Value Theorem. His aim, as he explained in the *Cours d'Analyse*, was to endow proof in analysis with the same level of rigour as proof in Euclid's geometry. However, the originality of his lectures within the programme of the École was not looked upon favourably, and he was criticised by the director of the École who warned Cauchy that 'It is the opinion of many persons that the instruction in pure mathematics is being carried too far at the École and such an uncalled-for extravagance is prejudicial to other branches of mathematics.'

Proof of Theorem F41 Consider the function

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$

By the Combination Rules for continuous functions and for differentiable functions, h is continuous on $[a, b]$ and differentiable on (a, b). Also,

$$\begin{aligned} h(a) &= f(a)(g(b) - g(a)) - g(a)(f(b) - f(a)) \\ &= f(a)g(b) - g(a)f(b) \end{aligned}$$

and

$$\begin{aligned} h(b) &= f(b)(g(b) - g(a)) - g(b)(f(b) - f(a)) \\ &= f(a)g(b) - g(a)f(b), \end{aligned}$$

so $h(a) = h(b)$.

Thus h satisfies the conditions of Rolle's Theorem on $[a, b]$, so there exists a point $c \in (a, b)$ for which

$$h'(c) = 0;$$

that is,

$$f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a)) = 0.$$

The two equations in the statement of Cauchy's Mean Value Theorem now follow by rearranging this equation. ■

5.2 L'Hôpital's Rule and its application

We are now in a position to prove the main result of this section.

Theorem F42 L'Hôpital's Rule

Let f and g be differentiable on an open interval I containing the point c, and suppose that $f(c) = g(c) = 0$. Then

$$\lim_{x \to c} \frac{f(x)}{g(x)} \text{ exists and equals } \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided that the latter limit exists.

Proof We assume that

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} \text{ exists and equals } l, \tag{10}$$

and we want to deduce that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = l. \tag{11}$$

We use the ε-δ definition of limit from Subsection 3.3 of Unit F1.

Let $\varepsilon > 0$. Then, by statement (10), there exists $\delta > 0$ such that

$$\left| \frac{f'(x)}{g'(x)} - l \right| < \varepsilon, \quad \text{for all } x \text{ with } 0 < |x - c| < \delta. \tag{12}$$

In particular, $g'(x) \neq 0$, for $0 < |x - c| < \delta$.

Suppose now that x is such that $0 < |x - c| < \delta$. If $g(x) = g(c)$, then g' must vanish at some point between c and x (by Rolle's Theorem), which we know is not the case. Therefore, $g(x) \neq g(c)$. Thus, by Cauchy's Mean Value Theorem, there exists some point d between c and x such that

$$\begin{aligned} \frac{f'(d)}{g'(d)} &= \frac{f(x) - f(c)}{g(x) - g(c)} \\ &= \frac{f(x)}{g(x)} \quad (\text{since } f(c) = g(c) = 0). \end{aligned}$$

Thus, by statement (12), we have

$$\left| \frac{f(x)}{g(x)} - l \right| = \left| \frac{f'(d)}{g'(d)} - l \right| < \varepsilon, \quad \text{for all } x \text{ with } 0 < |x - c| < \delta.$$

It follows that statement (11) is true, as required. ■

In the early 1690s the Marquis de l'Hôpital (1661–1704) contracted Johann Bernoulli (1667–1748) to teach him the recently published Leibnizian differential calculus. The result was the first textbook ever written on the calculus, l'Hôpital's *Analyse des infiniment petits, pour l'intelligence des lignes courbes* (*The analysis of the infinitely small, for the understanding of curved lines*), published in Paris in 1696. It contains what it is now known as l'Hôpital's rule, although l'Hôpital learnt the rule from Bernoulli. When l'Hôpital published his *Analyse* he acknowledged all the instruction he had received from Bernoulli but in such a way that Bernoulli took offence.


The Marquis de l'Hôpital

We can use l'Hôpital's Rule to evaluate the two complicated limits mentioned in the introduction to this section.

Worked Exercise F21

Prove that

$$\lim_{x\to\pi/2} \frac{\cos 3x}{\sin x - e^{\cos x}}$$

exists, and determine its value.

Solution

Let $I = \mathbb{R}$ and define

$$f(x) = \cos 3x \quad \text{and} \quad g(x) = \sin x - e^{\cos x} \quad (x \in \mathbb{R}).$$

Then f and g are differentiable, and $f(\pi/2) = g(\pi/2) = 0$; hence f and g satisfy the conditions of l'Hôpital's Rule at the point $x = \pi/2$.

Now the derivatives of f and g are

$$f'(x) = -3\sin 3x \quad \text{and} \quad g'(x) = \cos x + e^{\cos x}\sin x \quad (x \in \mathbb{R}).$$

Since $g'(\pi/2) = 1 \neq 0$, and f' and g' are both continuous, we deduce, by the Combination Rules for continuous functions, that

$$\begin{aligned}\lim_{x\to\pi/2} \frac{f'(x)}{g'(x)} &= \frac{f'(\pi/2)}{g'(\pi/2)}\\ &= \frac{(-3)\times(-1)}{1} = 3.\end{aligned}$$

Thus, by l'Hôpital's Rule, the required limit exists and

$$\lim_{x\to\pi/2} \frac{\cos 3x}{\sin x - e^{\cos x}} = 3.$$

The next example requires *two* applications of l'Hôpital's Rule.

Worked Exercise F22

Prove that

$$\lim_{x\to 0} \frac{x^2}{\cosh x - 1}$$

exists, and determine its value.

Solution

Let $I = \mathbb{R}$ and define

$$f(x) = x^2 \quad \text{and} \quad g(x) = \cosh x - 1 \quad (x \in \mathbb{R}).$$

Then f and g are differentiable, and $f(0) = g(0) = 0$; hence f and g satisfy the conditions of l'Hôpital's Rule at the point $x = 0$.

Now the derivatives of f and g are

$$f'(x) = 2x \quad \text{and} \quad g'(x) = \sinh x \quad (x \in \mathbb{R}).$$

Thus, by l'Hôpital's Rule, the required limit exists and equals

$$\lim_{x\to 0} \frac{2x}{\sinh x}, \tag{$*$}$$

provided that the limit $(*)$ exists.

Since $g'(0) = 0$, we cannot assert that

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \frac{f'(0)}{g'(0)}.$$

So we try to apply l'Hôpital's Rule a second time.

Both f' and g' are differentiable, and $f'(0) = g'(0) = 0$; hence f' and g' satisfy the conditions of l'Hôpital's Rule at the point $x = 0$. Now

$$f''(x) = 2 \quad \text{and} \quad g''(x) = \cosh x \quad (x \in \mathbb{R}).$$

Thus, by l'Hôpital's Rule, limit $(*)$ exists and equals

$$\lim_{x\to 0} \frac{2}{\cosh x},$$

provided that this limit exists. But the function cosh is continuous on $\mathbb{R}$, and $\cosh 0 = 1$. Thus, by the Quotient Rule for continuous functions,

$$\lim_{x\to 0} \frac{2}{\cosh x} = \frac{2}{1} = 2.$$

Working backwards, we conclude that limit $(*)$ exists and equals 2, so

$$\lim_{x\to 0} \frac{x^2}{\cosh x - 1} = 2.$$

Before asking you to apply l'Hôpital's Rule for yourself, we emphasise that you should *always* check carefully that its conditions hold, because a careless application can easily give an incorrect answer, as you will see in the following worked exercise!

Worked Exercise F23

Explain why the following proof is incorrect and find the correct value for the limit.

Claim (incorrect!)

$$\lim_{x \to 1} \frac{2x^2 - x - 1}{x^2 - x} = 2.$$

Proof (incorrect!) Let $I = \mathbb{R}$ and define

$$f(x) = 2x^2 - x - 1 \quad \text{and} \quad g(x) = x^2 - x \quad (x \in \mathbb{R}).$$

Then f and g are differentiable on $\mathbb{R}$, and

$$f(1) = g(1) = 0.$$

So, by l'Hôpital's Rule,

$$\begin{aligned} \lim_{x \to 1} \frac{f(x)}{g(x)} &= \lim_{x \to 1} \frac{f'(x)}{g'(x)} = \lim_{x \to 1} \frac{4x - 1}{2x - 1} \\ &= \lim_{x \to 1} \frac{f''(x)}{g''(x)} = \lim_{x \to 1} \frac{4}{2} = 2. \end{aligned}$$

■

Solution

The evaluation of the limit in the argument above involves *two* applications of l'Hôpital's Rule. The first application is valid, since f and g are differentiable on $\mathbb{R}$ and $f(1) = g(1) = 0$, so the conditions of l'Hôpital's Rule are satisfied. However, the conditions were not checked for the second application of the Rule, and in fact $f'(1) = 3 \neq 0$ and $g'(1) = 1 \neq 0$, so the conditions are *not* satisfied.

To evaluate the limit correctly, we should instead have concluded that, since f' and g' are both continuous, it follows from the Quotient Rule for continuous functions that

$$\lim_{x \to 1} \frac{f(x)}{g(x)} = \lim_{x \to 1} \frac{f'(x)}{g'(x)} = \frac{f'(1)}{g'(1)} = 3.$$

Exercise F32

Prove that the following limits exist, and evaluate them.

(a) $\lim_{x\to\pi} \dfrac{\sinh(x-\pi)}{\sin 3x}$ (b) $\lim_{x\to 0} \dfrac{(1+x)^{1/5}-(1-x)^{1/5}}{(1+2x)^{2/5}-(1-2x)^{2/5}}$

(c) $\lim_{x\to 0} \dfrac{\sin(x^2)}{1-\cos 4x}$ (d) $\lim_{x\to 0} \dfrac{\sin x - x\cos x}{x^3}$

Summary

In this unit you have met a formal definition of what it means for a function to be differentiable at a point and seen that, if a function f is differentiable at a point c, then the graph of f has a tangent at the point $(c, f(c))$. You have also seen how to use the definition to obtain the derivatives of some basic functions, and met many rules – in particular, the Glue Rule, the Combination Rules, the Composition Rule and the Inverse Function Rule – that enable us to show that many more functions are differentiable and to determine their derivatives.

You have studied the properties of functions that are differentiable on an interval, and seen that local minima and maxima occur at places where the derivative is zero. You have met Rolle's Theorem and seen how this is used in the proof of the Mean Value Theorem. This has useful corollaries such as the Increasing–Decreasing Theorem, which says that if the derivative of a function is positive then the function is increasing, while if the derivative is negative then the function is decreasing. This in turn enables us to use the second derivative test to determine whether a local extreme value is a maximum or minimum. Finally, you learnt how to use l'Hôpital's Rule to evaluate many limits of the form $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ where f and g are differentiable functions with $f(c) = g(c) = 0$.

Learning outcomes

After working through this unit, you should be able to:

- explain what is meant by a *differentiable function*, and understand the geometric significance of differentiability
- determine, using the definition, whether or not a function is differentiable at a point
- explain what is meant by a *second derivative* and a higher-order derivative
- explain what is meant by the *left derivative* and the *right derivative* of a function at a given point
- state and use the Glue Rule for differentiation
- use the table of standard derivatives
- use the rules for differentiation to prove the differentiability of a particular function and to calculate its derivative
- state and use the Local Extreme Value Theorem
- state and use Rolle's Theorem
- state and use the Mean Value Theorem
- state and use the Increasing–Decreasing Theorem and the Zero Derivative Theorem
- understand the statements of Cauchy's Mean Value Theorem and l'Hôpital's Rule
- use l'Hôpital's Rule to evaluate certain limits of the form $\lim_{x\to c} \frac{f(x)}{g(x)}$, where $f(c) = g(c) = 0$.

Table of standard derivatives

$f(x)$	$f'(x)$	Domain of f'
k	0	$\mathbb{R}$
x	1	$\mathbb{R}$
$x^n,\ n \in \mathbb{Z} - \{0\}$	nx^{n-1}	$\mathbb{R}$ or $\mathbb{R} - \{0\}$
$x^\alpha,\ \alpha \in \mathbb{R}$	$\alpha x^{\alpha-1}$	$\mathbb{R}^+$
$a^x,\ a > 0$	$a^x \log a$	$\mathbb{R}$
$\sin x$	$\cos x$	$\mathbb{R}$
$\cos x$	$-\sin x$	$\mathbb{R}$
$\tan x$	$\sec^2 x$	$\mathbb{R} - \left\{\left(n + \frac{1}{2}\right)\pi : n \in \mathbb{Z}\right\}$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$
$\sec x$	$\sec x \tan x$	$\mathbb{R} - \left\{\left(n + \frac{1}{2}\right)\pi : n \in \mathbb{Z}\right\}$
$\cot x$	$-\operatorname{cosec}^2 x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$
$\sin^{-1} x$	$1/\sqrt{1-x^2}$	$(-1, 1)$
$\cos^{-1} x$	$-1/\sqrt{1-x^2}$	$(-1, 1)$
$\tan^{-1} x$	$1/\left(1+x^2\right)$	$\mathbb{R}$
e^x	e^x	$\mathbb{R}$
$\log x$	$1/x$	$\mathbb{R}^+$
$\sinh x$	$\cosh x$	$\mathbb{R}$
$\cosh x$	$\sinh x$	$\mathbb{R}$
$\tanh x$	$\operatorname{sech}^2 x$	$\mathbb{R}$
$\sinh^{-1} x$	$1/\sqrt{1+x^2}$	$\mathbb{R}$
$\cosh^{-1} x$	$1/\sqrt{x^2-1}$	$(1, \infty)$
$\tanh^{-1} x$	$1/\left(1-x^2\right)$	$(-1, 1)$

Solutions to exercises

Solution to Exercise F15

(a) The difference quotient for f at c, where $c \neq 0$, is

$$\begin{aligned} Q(h) &= \frac{f(c+h) - f(c)}{h} \\ &= \frac{\dfrac{1}{c+h} - \dfrac{1}{c}}{h} \\ &= \frac{-1}{(c+h)c}, \quad \text{where } h \neq 0. \end{aligned}$$

Thus $Q(h) \to -1/c^2$ as $h \to 0$. Hence f is differentiable at c, with $f'(c) = -1/c^2$.

(b) The difference quotient for f at 0 is

$$\begin{aligned} Q(h) &= \frac{f(h) - f(0)}{h} \\ &= \frac{h^2 \cos(1/h) - 0}{h} \\ &= h\cos(1/h), \quad \text{where } h \neq 0. \end{aligned}$$

Now, $|\cos(1/h)| \leq 1$ for $h \neq 0$, so

$$|Q(h)| \leq |h|, \quad \text{for } h \neq 0.$$

Thus $Q(h) \to 0$ as $h \to 0$, by the Squeeze Rule for limits. Hence f is differentiable at 0, with $f'(0) = 0$.

(c) The difference quotient for f at 0 is

$$\begin{aligned} Q(h) &= \frac{f(h) - f(0)}{h} \\ &= \frac{|h| - 0}{h} \\ &= \begin{cases} 1, & h > 0, \\ -1, & h < 0. \end{cases} \end{aligned}$$

Now consider the two null sequences

$$h_n = \frac{1}{n} \quad \text{and} \quad k_n = -\frac{1}{n}, \quad n = 1, 2, \ldots.$$

These sequences have non-zero terms, and

$$Q(h_n) = 1 \to 1 \text{ as } n \to \infty,$$

but

$$Q(k_n) = -1 \to -1 \text{ as } n \to \infty.$$

Since these limits are different, f is not differentiable at 0.

(d) The difference quotient for f at 0 is

$$\begin{aligned} Q(h) &= \frac{f(h) - f(0)}{h} \\ &= \frac{|h|^{1/2}\sin(1/h) - 0}{h} \\ &= \begin{cases} \dfrac{\sin(1/h)}{h^{1/2}}, & h > 0, \\ -\dfrac{\sin(1/h)}{|h|^{1/2}}, & h < 0. \end{cases} \end{aligned}$$

Now consider the null sequence

$$h_n = \frac{1}{\left(2n + \frac{1}{2}\right)\pi}, \quad n = 1, 2, \ldots,$$

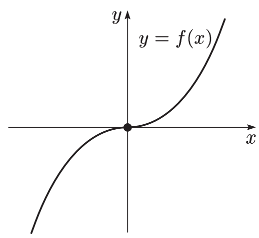
which has positive terms. This gives

$$\begin{aligned} Q(h_n) &= \frac{\sin(1/h_n)}{h_n^{1/2}} \\ &= \left(2n + \tfrac{1}{2}\right)^{1/2} \pi^{1/2} \sin\left(2n + \tfrac{1}{2}\right)\pi \\ &= \left(2n + \tfrac{1}{2}\right)^{1/2} \pi^{1/2} \to \infty \text{ as } n \to \infty. \end{aligned}$$

Hence f is not differentiable at 0.

Solution to Exercise F16

The graph of $y = f(x)$ is given below.



(This is included to aid your understanding – you do not need to sketch the graph as part of your solution.)

Let $I = \mathbb{R}$ and define

$$g(x) = -x^2 \quad (x \in \mathbb{R}) \quad \text{and} \quad h(x) = x^2 \quad (x \in \mathbb{R}).$$

Then

$$\begin{aligned} f(x) &= g(x), \quad \text{for } x < 0, \\ f(x) &= h(x), \quad \text{for } x > 0, \end{aligned} \qquad (*)$$

so condition 1 of the Glue Rule holds (with $c = 0$).

Furthermore, $f(0) = g(0) = h(0) = 0$, so condition 2 holds, and g and h are differentiable with

$$g'(x) = -2x \quad (x \in \mathbb{R}) \quad \text{and} \quad h'(x) = 2x \quad (x \in \mathbb{R}),$$

so condition 3 holds.

Since $g'(0) = h'(0) = 0$, it follows from the Glue Rule that f is differentiable at 0 and $f'(0) = 0$.

Also, by equations $(*)$,

$$f'(x) = g'(x) = -2x, \quad \text{for } x < 0,$$
$$f'(x) = h'(x) = 2x, \quad \text{for } x > 0,$$

since differentiability is a local property.

Hence f is differentiable (on $\mathbb{R}$), and

$$f'(x) = \begin{cases} -2x, & x < 0, \\ 0, & x = 0, \\ 2x, & x > 0. \end{cases}$$

Thus

$$f'(x) = 2|x| \quad (x \in \mathbb{R}).$$

Solution to Exercise F17

In each case we use the Combination Rules.

(a) $f'(x) = 7x^6 - 8x^3 + 9x^2 - 5 \quad (x \in \mathbb{R})$

(b) $$f'(x) = \frac{(x^3-1)2x - (x^2+1)3x^2}{(x^3-1)^2} = \frac{-x^4 - 3x^2 - 2x}{(x^3-1)^2} \quad (x \in \mathbb{R} - \{1\})$$

(c) $$f'(x) = \cos^2 x - \sin^2 x = \cos 2x \quad (x \in \mathbb{R})$$

(d) $$f'(x) = \frac{(3 + \sin x - 2\cos x)e^x - e^x(\cos x + 2\sin x)}{(3 + \sin x - 2\cos x)^2} = \frac{e^x(3 - \sin x - 3\cos x)}{(3 + \sin x - 2\cos x)^2} \quad (x \in \mathbb{R})$$

Solution to Exercise F18

We have

$$f'(x) = e^x + xe^x = e^x(1+x),$$
$$f''(x) = e^x(1+x) + e^x = e^x(2+x),$$
$$f^{(3)}(x) = e^x(2+x) + e^x = e^x(3+x).$$

Solution to Exercise F19

In each case, we use the Quotient Rule and the derivatives of sin and cos.

(a) $f(x) = \tan x = \sin x/\cos x$, so

$$f'(x) = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

on the domain of f.

(b) $f(x) = \operatorname{cosec} x = 1/\sin x$, so

$$f'(x) = -\frac{\cos x}{\sin^2 x} = -\operatorname{cosec} x \cot x$$

on the domain of f.

(c) $f(x) = \sec x = 1/\cos x$, so

$$f'(x) = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

on the domain of f.

(d) $f(x) = \cot x = \cos x/\sin x$, so

$$f'(x) = \frac{\sin x(-\sin x) - \cos x \cos x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

on the domain of f.

Solution to Exercise F20

In each case we use the Combination Rules.

(a) $f(x) = \sinh x = \frac{1}{2}(e^x - e^{-x})$, so

$$f'(x) = \tfrac{1}{2}(e^x + e^{-x}) = \cosh x.$$

(b) $f(x) = \cosh x = \frac{1}{2}(e^x + e^{-x})$, so

$$f'(x) = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x.$$

(c) $f(x) = \tanh x = \sinh x/\cosh x$, so

$$f'(x) = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.$$

Solution to Exercise F21

(a) $f(x) = \sinh(x^2)$, so

$$f'(x) = 2x\cosh(x^2).$$

(b) $f(x) = \sin(\sinh 2x)$, so

$$f'(x) = 2\cos(\sinh 2x)\cosh 2x.$$

(c) $f(x) = \sin\left(\dfrac{\cos 2x}{x^2}\right) \quad (x \in (0,\infty))$,

so on this interval

$$\begin{aligned} f'(x) &= \cos\left(\frac{\cos 2x}{x^2}\right)\left(\frac{x^2(-2\sin 2x) - 2x\cos 2x}{x^4}\right) \\ &= -\frac{2}{x^3}(x\sin 2x + \cos 2x)\cos\left(\frac{\cos 2x}{x^2}\right). \end{aligned}$$

Solution to Exercise F22

(a) The function

$$f(x) = \cos x \quad (x \in (0,\pi))$$

is continuous and strictly decreasing, and

$$f((0,\pi)) = (-1,1).$$

Also, f is differentiable on $(0,\pi)$, and its derivative $f'(x) = -\sin x$ is non-zero there.

Thus f satisfies the conditions of the Inverse Function Rule.

Hence f has an inverse function f^{-1} and $f^{-1} = \cos^{-1}$ is differentiable on $(-1,1)$. If $y = f(x) = \cos x$, then

$$(f^{-1})'(y) = \frac{1}{f'(x)} = -\frac{1}{\sin x}.$$

Since $\sin x > 0$ on $(0,\pi)$ and $\sin^2 x + \cos^2 x = 1$, it follows that

$$\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-y^2},$$

so

$$(f^{-1})'(y) = \frac{-1}{\sqrt{1-y^2}}.$$

Replacing the domain variable y by x, we obtain

$$(\cos^{-1})'(x) = \frac{-1}{\sqrt{1-x^2}} \quad (x \in (-1,1)).$$

(b) The function

$$f(x) = \sinh x \quad (x \in \mathbb{R})$$

is continuous and strictly increasing, and $f(\mathbb{R}) = \mathbb{R}$.

Also, f is differentiable on $\mathbb{R}$, and its derivative $f'(x) = \cosh x$ is non-zero there.

Thus f satisfies the conditions of the Inverse Function Rule.

Hence f has an inverse function f^{-1} and $f^{-1} = \sinh^{-1}$ is differentiable on $\mathbb{R}$. If $y = f(x) = \sinh x$, then

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{\cosh x}.$$

Since $\cosh x > 0$ on $\mathbb{R}$ and $\cosh^2 x = 1 + \sinh^2 x$, it follows that

$$\cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+y^2},$$

so

$$(f^{-1})'(y) = \frac{1}{\sqrt{1+y^2}}.$$

Replacing the domain variable y by x, we obtain

$$(\sinh^{-1})'(x) = \frac{1}{\sqrt{1+x^2}} \quad (x \in \mathbb{R}).$$

Solution to Exercise F23

(a) If $x_1 < x_2$, then $x_1^5 < x_2^5$, so $f(x_1) < f(x_2)$. Thus f is strictly increasing and continuous on $\mathbb{R}$, and $f(\mathbb{R}) = \mathbb{R}$. Also

$$f'(x) = 5x^4 + 1 \neq 0, \quad \text{for } x \in \mathbb{R}.$$

Thus f satisfies the conditions of the Inverse Function Rule. Hence f has an inverse function f^{-1} which is differentiable on $\mathbb{R}$.

(b) Now, $f(0) = -1$, $f(1) = 1$ and $f(-1) = -3$. Hence, by the Inverse Function Rule,

$$\begin{aligned} (f^{-1})'(-1) &= \frac{1}{f'(0)} = 1, \\ (f^{-1})'(1) &= \frac{1}{f'(1)} = \tfrac{1}{6}, \\ (f^{-1})'(-3) &= \frac{1}{f'(-1)} = \tfrac{1}{6}. \end{aligned}$$

Solution to Exercise F24

By definition,

$$f(x) = x^x = \exp(x \log x) \quad (x \in \mathbb{R}^+).$$

The functions $x \longmapsto x$ and $x \longmapsto \log x$ are differentiable on $\mathbb{R}^+$, and exp is differentiable on $\mathbb{R}$. It follows by the Product Rule and the Composition Rule that f is differentiable on $\mathbb{R}^+$, and that

$$\begin{aligned} f'(x) &= \exp(x \log x)\,(\log x + x \times (1/x)) \\ &= x^x(\log x + 1) \quad (x \in \mathbb{R}^+). \end{aligned}$$

Solution to Exercise F25

Since the functions sin and cos are continuous and differentiable on $\mathbb{R}$, so also is f, by the Combination Rules.

Now,

$$f'(x) = 2\sin x \cos x - \sin x = \sin x(2\cos x - 1);$$

thus f' vanishes on $(0, \pi/2)$ when $\cos x = \frac{1}{2}$, that is, when $x = \pi/3$.

Since $f(0) = 1$, $f(\pi/2) = 1$ and

$$f(\pi/3) = (\sqrt{3}/2)^2 + \tfrac{1}{2} = \tfrac{3}{4} + \tfrac{1}{2} = \tfrac{5}{4},$$

it follows that on $[0, \pi/2]$:

the minimum of f is 1, occurring at $x = 0$ and $\pi/2$;
the maximum of f is $\frac{5}{4}$, occurring at $x = \pi/3$.

Solution to Exercise F26

Since f is a polynomial function, f is continuous on $[1, 3]$ and differentiable on $(1, 3)$. Also, $f(1) = 2$ and $f(3) = 2$, so $f(1) = f(3)$.

Thus f satisfies the conditions of Rolle's Theorem on $[1, 3]$, so there exists c in $(1, 3)$ such that $f'(c) = 0$.

(In fact, since

$$f'(x) = 4x^3 - 12x^2 + 6x = 2x(2x^2 - 6x + 3),$$

and $2x^2 - 6x + 3 = 0$ for $x = \frac{1}{2}(3 \pm \sqrt{3})$, we have $c = \frac{1}{2}(3 + \sqrt{3}) \simeq 2.37$.)

Solution to Exercise F27

(a) No: f is not defined at $\pi/2$.

(b) No: f is not differentiable at 1.

(c) Yes: all the conditions are satisfied.

(d) No: $f(0) \neq f(\pi/2)$.

Solution to Exercise F28

The function $f(x) = xe^x$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$ by the Product Rule. Thus f satisfies the conditions of the Mean Value Theorem on $[0, 2]$.

Now,

$$\frac{f(2) - f(0)}{2 - 0} = \frac{2e^2 - 0}{2} = e^2.$$

Thus, by the Mean Value Theorem, there exists a point c in $(0, 2)$ such that $f'(c) = e^2$.

Solution to Exercise F29

(a) The function f is continuous on I and differentiable on the interior of I and so we can apply the Increasing–Decreasing Theorem. We have $f'(x) = 4x^{1/3} - 4 = 4(x^{1/3} - 1)$. Thus $f'(x) > 0$ for $x \in (1, \infty)$, so f is strictly increasing on $[1, \infty)$, by the strict inequalities version of the Increasing–Decreasing Theorem.

(b) The function f is continuous on I and differentiable on the interior of I and so we can apply the Increasing–Decreasing Theorem. We have $f'(x) = 1 - 1/x = (x - 1)/x$. Thus $f'(x) < 0$ for $x \in (0, 1)$, so f is strictly decreasing on $(0, 1]$, by the strict inequalities version of the Increasing–Decreasing Theorem.

Solution to Exercise F30

(a) We have

$$f'(x) = 3x^2 - 6x = 3x(x - 2).$$

Thus $f'(x) = 0$ for $x = 0$ and 2, so the required values of c are 0 and 2.

(b) We have

$$f''(x) = 6x - 6,$$

so

$$f''(0) = -6 < 0 \quad \text{and} \quad f''(2) = 6 > 0.$$

Also, $f(0) = 1$ and $f(2) = -3$.

Since f is a twice-differentiable function defined on $\mathbb{R}$ and f'' is continuous on $\mathbb{R}$, it follows from the Second Derivative Test that f has a local maximum of 1 at $x = 0$ and a local minimum of -3 at $x = 2$.

Solution to Exercise F31

In each case we follow the steps in Strategy F7.

(a) 1. Let

$$f(x) = x - \sin x \quad (x \in [0, \infty)).$$

Then f is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$.

2. We have

$$f'(x) = 1 - \cos x \geq 0, \quad \text{for } x \in (0, \infty),$$

and $f(0) = 0$.

Thus f is increasing on $[0, \infty)$, by the Increasing–Decreasing Theorem, so

$$f(x) \geq f(0) = 0, \quad \text{for } x \in [0, \infty).$$

Hence

$$\sin x \leq x, \quad \text{for } x \in [0, \infty).$$

(b) 1. Let

$$f(x) = \tfrac{2}{3}x + \tfrac{1}{3} - x^{2/3} \quad (x \in [0, 1]).$$

Then f is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

2. We have

$$\begin{aligned} f'(x) &= \tfrac{2}{3} - \tfrac{2}{3}x^{-1/3} \\ &= \tfrac{2}{3}(1 - x^{-1/3}) < 0, \quad \text{for } x \in (0, 1), \end{aligned}$$

and $f(1) = \frac{2}{3} + \frac{1}{3} - 1 = 0$.

Thus f is decreasing on $[0, 1]$, by the Increasing–Decreasing Theorem, so

$$f(x) \geq f(1) = 0, \quad \text{for } x \in [0, 1].$$

Hence

$$\tfrac{2}{3}x + \tfrac{1}{3} \geq x^{2/3}, \quad \text{for } x \in [0, 1].$$

Solution to Exercise F32

(a) Let $I = \mathbb{R}$ and define

$$f(x) = \sinh(x - \pi) \quad (x \in \mathbb{R})$$

and

$$g(x) = \sin 3x \quad (x \in \mathbb{R}).$$

Then f and g are differentiable on $\mathbb{R}$, and

$$f(\pi) = g(\pi) = 0.$$

Thus f and g satisfy the conditions of l'Hôpital's Rule at the point $x = 0$.

Now,

$$f'(x) = \cosh(x - \pi) \quad \text{and} \quad g'(x) = 3\cos 3x.$$

Since $g'(\pi) = -3 \neq 0$, and f' and g' are both continuous, we deduce, by the Combination Rules for continuous functions, that

$$\lim_{x \to \pi} \frac{f'(x)}{g'(x)} = \frac{f'(\pi)}{g'(\pi)} = \frac{1}{-3} = -\tfrac{1}{3}.$$

Thus, by l'Hôpital's Rule, the required limit exists and equals $-\frac{1}{3}$.

(b) Let $I = \left(-\frac{1}{2}, \frac{1}{2}\right)$, say, and define

$$f(x) = (1 + x)^{1/5} - (1 - x)^{1/5}$$

and

$$g(x) = (1 + 2x)^{2/5} - (1 - 2x)^{2/5}$$

on I. (The only requirements when selecting the open interval I are that it must contain 0 and lie in the domains of both f and g; that is, all $x \in I$ must satisfy $1 - 2x \geq 0$ and $1 + 2x \geq 0$.)

Then f and g are differentiable on I and

$$f(0) = g(0) = 0.$$

Thus f and g satisfy the conditions of l'Hôpital's Rule at the point $x = 0$.

Now,

$$f'(x) = \tfrac{1}{5}(1 + x)^{-4/5} + \tfrac{1}{5}(1 - x)^{-4/5}$$

and

$$g'(x) = \tfrac{4}{5}(1 + 2x)^{-3/5} + \tfrac{4}{5}(1 - 2x)^{-3/5}.$$

Since $g'(0) = \frac{4}{5} + \frac{4}{5} \neq 0$, and f' and g' are both continuous, we deduce, by the Combination Rules for continuous functions, that

$$\lim_{x\to 0} \frac{f'(x)}{g(x)} = \frac{f'(0)}{g'(0)} = \frac{\frac{1}{5}+\frac{1}{5}}{\frac{4}{5}+\frac{4}{5}} = \tfrac{1}{4}.$$

Thus, by l'Hôpital's Rule, the required limit exists and equals $\frac{1}{4}$.

(c) Let $I = \mathbb{R}$ and define

$$f(x) = \sin(x^2) \quad \text{and} \quad g(x) = 1 - \cos 4x \quad (x \in \mathbb{R}).$$

Then f and g are differentiable on $\mathbb{R}$, and

$$f(0) = g(0) = 0.$$

Thus f and g satisfy the conditions of l'Hôpital's Rule at the point $x = 0$.

Now,

$$f'(x) = 2x\cos(x^2) \quad \text{and} \quad g'(x) = 4\sin 4x.$$

Thus, by l'Hôpital's Rule, the required limit exists and equals

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)}, \qquad (*1)$$

provided that limit $(*1)$ exists. Here $f'(0) = g'(0) = 0$, so we cannot apply l'Hôpital's Rule at this stage. However, f' and g' are differentiable on $\mathbb{R}$, so f' and g' satisfy the conditions of l'Hôpital's Rule at the point $x = 0$.

Now,

$$f''(x) = -(2x)^2\sin(x^2) + 2\cos(x^2)$$

and

$$g''(x) = 16\cos 4x.$$

Thus, by l'Hôpital's Rule, the required limit exists and equals

$$\lim_{x\to 0} \frac{f''(x)}{g''(x)}, \qquad (*2)$$

provided that limit $(*2)$ exists.

Since $g''(0) = 16 \neq 0$, and f'' and g'' are both continuous, we deduce, by the Combination Rules for continuous functions, that

$$\lim_{x\to 0} \frac{f''(x)}{g''(x)} = \frac{f''(0)}{g''(0)} = \frac{2}{16} = \tfrac{1}{8}.$$

Hence limit $(*2)$ exists and equals $\frac{1}{8}$.

Thus limit $(*1)$ exists and equals $\frac{1}{8}$, so the required limit also exists and equals $\frac{1}{8}$.

(d) Let $I = \mathbb{R}$ and define

$$f(x) = \sin x - x\cos x \quad \text{and} \quad g(x) = x^3 \quad (x \in \mathbb{R}).$$

Then f and g are differentiable on $\mathbb{R}$, and $f(0) = g(0) = 0$. Thus f and g satisfy the conditions of l'Hôpital's Rule at the point $x = 0$.

Now,

$$f'(x) = x\sin x \quad \text{and} \quad g'(x) = 3x^2.$$

Thus, by l'Hôpital's Rule, the required limit exists and equals

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)},$$

provided that this limit exists. But

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{x\sin x}{3x^2} = \frac{1}{3}\lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{3}.$$

Hence, the required limit exists and equals $\frac{1}{3}$.